Introduction to Differential Equation

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Contents

Chapter 1

1st Order Differential Equation

1.1 Differential Equations

Many natural laws in Physics, Chemistry, and Engineering are expressed by means of differential equations. Among many natural laws, we treat Newton's 2nd law, Newton's cooling law, Kirchhoff's 2nd law, Fourier's law of heat transfer, wave equation, chemical equation.

A differential equation is a relation between an unknown function and derivatives of the unknown function. Differential equations we treat are the ones that can be solved by quadrature.

A relation among an independent variable x, and a function $y(x)$, $y', y'', \ldots, y^{(n)}$ expressed in

$$F(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0$$

is called the **ordinary differential equation** in y.

We give some examples of differential equations. Note that $y' \equiv \dfrac{dy}{dx},\ y'' = \dfrac{d^2y}{dx^2}$.

$$\begin{array}{lll} (a)\ y' = x^3 & (b)\ y' = 2x^2 + y & (c)\ xy'' + y' = 0 \\ (d)\ (y')^2 = x & (e)\ y'' + xyy' = x & (f)\ xy'' - \sin y = x^3 \end{array}$$

Order

The **order** of the differential equation is the highest order derivative present in the differential equation.

Differential equations $(a), (b), (d)$ are the 1st order and the rest is 2nd order. The natural laws we mentioned above are formulated as the 1st order or the 2nd order differential equation. In this book, we deal mainly with the 1st and the 2nd order differential equations.

Linear

A differential equation is called **linear** if the equation is expressed as an unknown function and it's derivatives. Otherwise it is called **non linear**. Thus, nth order linear differential equation is expressed in the following form.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = b(x),$$

NOTE From the definition above, differential equations $(a) - (c)$ are linear, $(d) - (f)$ are non linear.

Solutions of Differential Equations

If a function saitisfies the given differential equation on some interval, then the function is called the **solution** of the differential equation. To find a solution of the given differential equation, we say solve the differential equation. A solution may not be in the form of $y = f(x)$. It sometimes given by implicit function such as $G(x, y) = 0$.

Example 1.1 Show the functions e^{2x}, ce^{2x} are solutions of $y' = 2y$. Show the functions $e^x, e^x + c$ $(c \neq 0)$ are not solution of $y' = 2y$.

SOLUTION The derivatives of e^{2x} and ce^{2x} are $2e^{2x}$ and $2ce^{2x}$ respectively. Thus these satisfy $y' = 2y$. Since $(e^x)' = e^x \not\equiv 2e^x, (e^x + c)' = e^x \not\equiv 2e^x + c$ $(c \neq 0)$. Thus they are not the solutions of $y' = 2y$ ■

Example 1.2 Show that $y_1 = x^2 + x^3$ and $y_2 = x^3$ are solutions of $y' = 2(y/x) + x^2$.

SOLUTION $y_1' = 2x + 3x^2$ and $2(y_1/x) + x^2 = (2x^2 + x^3)/x + x^2 = 2x + 3x^2$. Thus, $y_1(x)$ is a solution. Similarly, $y_2' = 3x^2$ and $2(y_2/x) + x^2 = 2x^3/x + x^2 = 3x^2$. Thus, $y_2(x)$ is a solution ■

As you can see from this example, solutions of a differential equation may not be unique. In general, any nth order differential equation has n different solutions.

General solution

We say a linear combination of n independent solutions of n constants the **general solution**.

Particular solution

Any solution derived from substituting a particular value into the general solution is called the **particular solution**.

For example, $y = ce^x$ is the general solution of $y' = y$ and $y = e^x$ is the particular solution.

Singular solution

A **singular solution** is a solution which can not be derived from substituting a value into the general solution.

Example 1.3 Show that $\frac{y}{1-y} = ce^x$ is the general solution of $y' = y(1-y)$. Then show no matter what value you choose for c, $y = 1$ can not be obtained. Show $y = 1$ is the singular solution.

SOLUTION Differentiate $\frac{y}{1-y} = ce^x$ with respect to x. Then

$$\frac{1}{(1-y)^2}y' = ce^x$$

Since $ce^x = \frac{y}{1-y}$, we have

$$\frac{1}{(1-y)^2}y' = \frac{y}{1-y}$$

Then

$$y' = \frac{y}{1-y}(1-y)^2 = y(1-y)$$

Thus, $\frac{y}{1-y} = ce^x$ is the general solution. Now $y = 1$ can not be derived from the general solution. Thus $y = 1$ is the singular solution.

Solution curves

Note that the general solution of a differential equation forms curves. A each curve is called the **solution curve** or the **integral curve**.

Initial conditions, Initial value problem

In a differential equation $F(x, y, y', \cdots, y^{(n)}) = 0$, **initial conditions** are values of the solution and its derivatives at specific points such as

$$y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

The problem of finding a solution to the differential equation with the initial condition is called the **initial value problem**

Boundary condition, Boundary value problem

In a differential equation $F(x, y, y', \cdots, y^{(n)}) = 0$ on $[a, b]$, the condition must be satisfied by the solution or the derivatives of the solution is called the **boundary conditions**. The problem of finding a solution to the differential equation with the boundary conditions is called the **boundary value problem**

1.1.1 Exercise

1. Show that $e^x, xe^x, c_1e^x + c_2xe^x$ are solutions of the differential equation $y'' - 2y' + y = 0$. For $y(0) = 1,\ y'(0) = -1$, find c_1, c_2.

2. $\sin x, \cos x$ and their linear combinations are solutions to the differential equation $y'' + y = 0$. What can you say about the vector space spanned by $\sin x, \cos x$.

1.2 Separation of variables

The first order differential equation is expressed as

$$y' = f(x, y)$$

or

$$M(x, y)dx + N(x, y)dy = 0$$

The latter equation is called the **total differential equation**. Now consider

$$y' = -\frac{y}{x}, \ xdx + ydy = 0$$

Both equations are the same except at $x = 0$. Thus we formally treat both equations as the same equation.

Separable Differential Equations

If the differential equation $y' = f(x, y)$ is expressed in the form

$$y' = F(x)G(y),$$

then we say the differential equation is **separable**. To solve this equation,

$$y'(x) = F(x)G(y(x))$$

we express $y'(x)$ as $\frac{dy}{dx}$, and then write

$$\frac{dy}{G(y)} = F(x)dx$$

Now integrate both sides with respect to x to obtain

$$\int \frac{dy}{G(y)} dx = \int F(x)dx.$$

Example 1.4 Find the general solution of $y' = \frac{xy}{y-1}$.

SOLUTION We can rewrite this differential equation as

$$\frac{(y-1)dy}{y} = xdx$$

This is separable and integrate both sides to obtain

$$y - \log|y| = \frac{x^2}{2} + c$$

This is an implicit solution. Now let $c_1 = \log c$. Then

$$y - \log(c_1 y)^2 = x^2$$

Note that $y(x) \equiv 0$ is a solution. But no matter how you choose c_1, it is impossible to obtain $y = 0$. Thus $y = 0$ is a singular solution. As we noted above, when we are asked to find the general solution, we solve the differential equation by quadrature. Thus do not worry about a singular solution.

Newton's law of cooling

Let T_1 and T_2 be the temparature of two objects facing each other. Then the heat transfers from warmer body to cooler body in the time dt is given by dQ and

$$dQ = \kappa(T_1 - T_2)dt. \text{ where } \kappa \text{ is constant}$$

Example 1.5 Let the iron ball be heated to $100^\circ c$. Then submerge the iron ball into the water whose temparature is kept at $10^\circ c$. After 4 minutes, the temparature of the iron ball is $60^\circ c$. Find the time when the temparature of the iron ball is $20^\circ c$.

SOLUTION We formulate this problem by using Newton's law of cooling. Then

$$\frac{dT}{dt} = -\kappa(T - 10),\ T(0) = 100,\ T(4) = 60,$$

where $T(t)$ is a temparature of the iron ball after t minutes later. Since this differential equation is separable, we obtain

$$\frac{dT}{T - 10} = -\kappa dt$$

Integrating both sides with respect to t. Then

$$\log|T - 10| = -\kappa t + c$$

and

$$T(t) = ce^{-kt} + 10$$

Now by the boundary conditions, $T(0) = 100$ and $T(4) = 60$, we have

$$c = 90\text{and } k = \frac{1}{4}\log\frac{9}{5} \doteq \frac{1}{4}(2.1972 - 1.6094) \doteq 0.147$$

Thus,

$$T(t) \doteq 90e^{-0.147t} + 10.$$

Now we find t_0 satisfying $T(t) = 20$.

$$20 \doteq 90e^{-0.147t_0} + 10$$

and

$$t_0 \doteq \frac{\log 9}{0.147} \doteq 14.9 \text{ min} \blacksquare$$

1.2.1 Exercise

1. Find the general solution of the following differential equations.
 (a) $(\sin x)y' + (\cos x)y = 0$ (b) $y' = e^{x+y}$
 (c) $y' = \frac{1-y^2}{1-x^2}$ (d) $y' = (1 + 2x)(1 + y^2)$
2. Solve the following initial value problem
 (a) $(1 + e^x)y' = y,\ y(0) = 1$ (b) $y' + y\sin x = 0,\ y(\pi) = 3$
 (c) $xyy' - y^2 = 1,\ y(2) = 1$ (d) $y' = \frac{x(y^2-1)}{(x-1)y^3},\ y(2) = 2$

3. The body heated at $70^\circ c$ is put outside whose temparature is $20^\circ c$. Suppose that the temparature becomes $50^\circ c$ after 15 minutes
(a) Find the temparature of the body after 30 minutes.
(b) Find the time when the body temparature becomes $32^\circ c$.

1.3 Homogeneous Differential Equations

Using substitution, we often can transform a differential equation into simpler form. Here we consider how to transform the homogeneous differential equation into separable differential equation.

Homogeneous

If a 1st order differential equation $y' = f(x, y)$ is put into the form

$$y' = f(\frac{y}{x}),$$

then we say the differential equation is **homogeneous**. Now put

$$v = \frac{y}{x}$$

Then $y = vx$ and $y' = v'x + v$. Substitute these into the above equation, we have

$$v'x + v = f(v)$$

and

$$\frac{dv}{f(v) - v} = \frac{1}{x}dx$$

This is separable. Thus integrate both sides

$$\int \frac{dv}{f(v) - v} = \log x + c$$

Finally, substitute $v = \frac{y}{x}$ to get the general solution.

Homogeneous functions

If a function $M(x, y)$ satisfies $M(tx, ty) = t^n M(x, y)$, then we say $M(x, y)$ is the **homogeneous function** of degree n.

How to find the given differential equation is homogeneous or not

Given

$$M(x,y)dx + N(x,y)dy = 0, \; y' = -\frac{M(x,y)}{N(x,y)}$$

if $M(x,y)$ and $N(x,y)$ are the homogeneous functions of the same degree, then the differential equation is homogeneous.

Example 1.6 Find the general solution of $y' = \frac{x-y}{x+y}$.

SOLUTION $M(x,y), N(x,y)$ are the homogeneous functions of the same degree 1. Divide the numerator and the denominator by x. Then

$$y' = \frac{1-(y/x)}{1+(y/x)} = f(\frac{y}{x})$$

Let $y = vx$. Then $y' = v'x + v$ and

$$v'x + v = \frac{1-v}{1+v}$$

Simplifying

$$v'x = \frac{1-v}{1+v} - v = \frac{1-v-v-v^2}{1+v}$$

or

$$\frac{1+v}{v^2+2v-1}dv = -\frac{1}{x}dx$$

Integrate both sides

$$\frac{1}{2}\log|v^2+2v-1| = -\log x + c,$$

Multiplying 2 to both sides to get

$$\log|v^2+2v-1| = -\log x^2 + c.$$

Now take e as base to get

$$e^{\log|v^2+2v-1|} = e^{-\log x^2}c$$

$$(v^2+2v-1)x^2 = c.$$

Finally replace $v = y/x$ to obtain the general solution

$$y^2 + 2xy - x^2 = c \blacksquare$$

Example 1.7 Find the general solution to $y' = \frac{x-y+1}{x+y-1}$.

SOLUTION This is not homogeneou. But once we can get rid of the constant term, it becomes homogeneous. The intersetion of two lines

$$x - y + 1 = 0, x + y - 1 = 0$$

is $(0,1)$. we move the axis so that $(0,1)$ is the origin. Let $X = 0, Y = y - 1$. Then $x = X + 0, y = Y + 1$ and

$$\frac{dy}{dx} = \frac{dy}{dY}\frac{dY}{dX}\frac{dX}{dx} = \frac{dY}{dX}$$

Thus we have

$$\frac{dY}{dX} = \frac{X-Y}{X+Y}$$

This differential equation is solved in example 1.6. Thus the general solution is

$$Y^2 + 2XY - X^2 = c.$$

Since $Y = y-1, X = x$, we have

$$(y-1)^2 + 2(y-1)x - x^2 = c \blacksquare$$

Example 1.8 Find the general solution of $y' = \frac{x+y-1}{x+y+1}$.

SOLUTION Let $u = x+y$. Then $y = u - x$ and $y' = u' - 1$. Put this back into the original differential equation to obtain

$$u' - 1 = \frac{u-1}{u+1},$$

$$u' = \frac{u-1}{u+1} + 1 = \frac{u-1+u+1}{u+1} = \frac{2u}{u+1},$$

$$\frac{u+1}{u} du = 2dx.$$

Integrate both sides

$$u + \log|u| = 2x + c.$$

Then the general solution is

$$y - x + \log|x+y| = c \blacksquare$$

1.3.1 Exercise

1. Find the general solution of the following differential equations.
$(a)\ y' = \frac{xy}{x^2+y^2}$ $(b)\ y' = \frac{xy}{(x+2y)^2}$
$(c)\ y' = \frac{x^2+2xy-4y^2}{x^2-8xy-4y^2}$ $(d)\ (x^2 - y^2 e^{\frac{x}{y}})y' = xy$
$(e)\ y' = \frac{x+2y-1}{x+2y+7}$ $(f)\ y' = \frac{x-y+8}{y-3x+2}$
2. Solve the initial value problems.
$(a)\ (y - \sqrt{x^2+y^2})dx - xdy = 0,\ y(\sqrt{3}) = 1$
$(b)\ (y^3 - x^3)dx - xy^2 dy = 0,\ y(1) = 2$
3. Give a reason why the example 1.8 can not be solved by the technique shown in example 1.7.

1.4 Exact differential equations

Total differentials

The total differential of $u(x,y)$ is

$$du = u_x dx + u_y dy$$

Thus if we know the values of u_x, u_y, then we can find du. On the other hand, if we know the total differential du of a function u, then we can determine the function u.

Example 1.9 The total differential of $u(x, y)$ is given by

$$du = (2xy - \cos x dx + (x^2 - 1)dy$$

Find $u(x, y)$.

SOLUTION Since $du = u_x dx + u_y dy$, we have

$$u_x = 2xy - \cos x,\ \ u_y = x^2 - 1.$$

Integrate the first equation by x to obtain

$$u(x, y) = \int (2xy - \cos x)dx = x^2 y - \sin x + c(y)$$

Note that $c(y)$ is an arbitrary function of y. Now differentiate with respect ot y,

$$u_y = x^2 + c'(y)$$

Now u_y obtained and the u_y given above must be the same. Thus

$$u_y = x^2 + c'(y) = x^2 - 1.$$

From this we have $c'(y) = -1$ and $c(y) = -y + C$. Therefore,

$$u(x, y) = x^2 y - \sin x - y + C \blacksquare$$

General solution of exact differential equations

Definition 1.1 The left-hand side of the 1st order differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is the same as the differential du of some function u. Then the general solution is given by

$$u(x, y) = c$$

Example 1.10 Find the general solution of $(2xy - \cos x dx + (x^2 - 1)dy = 0$.

SOLUTION In the example above, we found the function u so that du is equal to the left-hand side of equation. Thus the general solution is

$$x^2 y - \sin x - y = c \blacksquare$$

Necessary and sufficient conditions for exact differential equations

Theorem 1.1 Let $M(x, y)$ and $N(x, y)$ be the class C^1 on $\{(x, y) : a < x < b, c < y < d\}$. Then the followings are equivalent
(1) $M(x, y)dx + N(x, y)dy = 0$ is exact
(2) $M_y = N_x$

Proof (1) $\Rightarrow$ (2)
If the differential equation is exact, then there exists $u(x,y)$ satisying $du = M(x,y)dx + N(x,y)dy$. Thus we have

$$M = u_x, N = u_y$$

Now partially differentiate M with respect to y and partially differentiate N with respect to x. Then

$$M_y = u_{xy}, N_x = u_{yx}$$

Since M_y, N_x are continuous, u_{xy}, u_{yx} are also continuous.Thus by **Schwarz lemma**, $u_{xy} = u_{yx}$ and $M_y = N_x$.
(2) $\Rightarrow$ (1)
Let (x_0, y_0) be a point in the domain of M, N. Consider

$$u(x,y) = \int_{x_0}^{x} M(x,y)dx + \int_{y_0}^{y} N(x_0,y)dy$$

Then $u_x(x,y) = M(x,y)$. Since $M_y = N_x$,

$$\begin{aligned} u_y(x,y) &= \int_{x_0}^{x} M_y(x,y)dx + N(x_0,y) = \int_{x_0}^{x} N_x(x,y)dx + N(x_0,y) \\ &= (N(x,y) - N(x_0,y)) + N(x_0,y) = N(x,y) \end{aligned}$$

Therefore,

$$du = M(x,y)dx + N(x,y)dy$$

and $M(x,y)dx + N(x,y)dy = 0$ is exact differential equation. ■

In the proof above, the general solution of $u(x,y)$ is given.

$$u(x,y) = \int_{x_0}^{x} M(x,y)dx + \int_{y_0}^{y} N(x_0,y)dy = c$$

Example 1.11 Find the general solution of the following differential equation

$$(y\cos x - \sin y)dx + (\sin x - x\cos y)dy = 0.$$

SOLUTION

$$M_y = \frac{\partial}{\partial y}(y\cos x - \sin y) = \cos x - \cos y,$$

$$N_x = \frac{\partial}{\partial x}(\sin x - x\cos y) = \cos x - \cos y$$

Thus it is exact differential equation. Then set $(x_0, y_0) = (0,0)$ to have

$$\begin{aligned} u(x,y) &= \int_{0}^{x} M(x,y)dx + \int_{0}^{y} N(0,y)dy \\ &= \int_{0}^{x} (y\cos x - \sin y)dx + \int_{0}^{y} 0dy \\ &= y\sin x - x\sin y \end{aligned}$$

Thus

$$y\sin x - x\sin y = c \;■$$

Example 1.12 Solve the initial value problem.

$$(2xy - 3y)dx + (4y^3 + x^2 - 3x + 4)dy = 0, \ y(0) = 1.$$

SOLUTION Note that $M_y = 2x - 3 = N_x$,

$$\begin{aligned} u(x,y) &= \int_0^x M(x,y)dx + \int_0^y N(0,y)dy \\ &= \int_0^x (2xy - 3y)dx + \int_0^y (4y^3 + 4)dy \\ &= x^2y - 3xy + y^4 + 4y. \end{aligned}$$

Thus the general solution is

$$x^2y - 3xy + y^4 + 4y = c$$

Since $y(0) = 1$, we have $c = 5$ ■

Instead of using the formula $u(x,y) = \int_{x_0}^{x} M(x,y)dx + \int_{y_0}^{y} N(x_0,y)dy$, we introduce a simpler method called **grouping method**.

Example 1.13 Find the general solution of $(2x + 3y)dx + (3x + y^2 + 3)dy = 0$.

SOLUTION Note that $M_y = 3 = N_x$. Thus it is exact. Now we write $M(x,y), N(x,y)$ as

$$\underbrace{2xdx}_{\text{function of } x} + \underbrace{(3ydx + 3xdy)}_{\text{function of } x,y} + \underbrace{(y^2+3)dy}_{\text{function of } y} = 0$$

Now we write these by using the total differentiation.

$$d(x^2) + d(3xy) + d(\frac{y^3}{3} + 3y) = d(c).$$

Then,

$$x^2 + 3xy + \frac{y^3}{3} + 3y = c \ ■$$

1.4.1 Exercise

1. Determine the following differential equations are exact or not. If exact, find the general solution.
(a) $(x^2 + y^2)dx + 2xydy = 0$ (b) $(ye^{xy} + 2xy)dx + (xe^{xy} + x^2)dy = 0$
(c) $(1 + xy^2)dx + (x^2y + y)dy = 0$ (d) $(y^2 - x^2)dx + 2xydy = 0$

2. Solve the initial value problem.
(a) $x^2dx + ye^ydy = 0, \ y(0) = 1$
(b) $(e^xy + \sin y)dx + (e^x + x\cos y)dy = 0, \ y(0) = 1$
(c) $(\cos x \sin x - xy^2)dx - y(x^2 - 1)dy = 0, \ y(0) = 2$

1.5 Integrating Factor

Integrating factor

Suppose that $M(x,y)dx + N(x,y)dy = 0$ is not exact differential equation. If

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact differential equation provided $\mu(x,y) \not\equiv 0$, then $\mu(x,y)$ is called the **integrating factor**.

In general, an integrating factor is not unique. For example, a function x^{-2} and y are the integrating factor of the following differential equation

$$2xdx + \frac{x^2}{y}dy = 0.$$

Now the question is how to find a integratin factor. Suppose that

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact. Then

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

Rewrite this to get

$$N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = \mu(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$$

This partial differential equation is hard to solve. Thus we concentrate on the special cases.

$\mu(x,y) = \mu(x)$

For $\mu(x,y) = \mu(x)$, $M\frac{\partial \mu}{\partial y} = 0$. Thus,

$$N\frac{\partial \mu}{\partial x} = \mu(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$$

Then

$$\frac{1}{\mu}d\mu = \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})dx.$$

Note that if

$$\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$$

is a function of x only, then the integrating factor is given by

$$\mu(x) = \exp\{\int \frac{1}{N}[M_y - N_x]dx\}$$

$\mu(x, y) = \mu(y)$

For $\mu(x, y) = \mu(y)$, $N\frac{\partial \mu}{\partial x} = 0$. Thus,

$$-M\frac{\partial \mu}{\partial y} = \mu(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$$

Then

$$\frac{1}{\mu}d\mu = -\frac{1}{M}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})dy.$$

Note that if

$$\frac{1}{M}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$$

is a function of y only, then the integrating factor is given by

$$\mu(y) = \exp\{-\int \frac{1}{M}[M_y - N_x]dy\}$$

Note that since we are looking for one integrating factor, we ignore the constant.

Example 1.14 Find the general solution of $(2x^2 - y)dx + (x^2y + x)dy = 0$.

SOLUTION Note that $M_y = -1, N_x = 2xy + 1$. Thus, this differential equation is not exact. Then we calculate $(1/N)[M_y - N_x]$.

$$\frac{1}{N}[M_y - N_x] = \frac{1}{x^2y + x}[-1 - (2xy + 1)] = -\frac{2(xy + 1)}{x(xy + 1)} = -\frac{2}{x}.$$

This is a function of x only. Then the integrating factor is

$$\mu = \exp(-\int \frac{2}{x}dx) = \exp(-2\log|x|) = \frac{1}{x^2}$$

Now multiply this to the original differential equation to obtain

$$(2 - \frac{y}{x^2})dx + (y + \frac{1}{x})dy = 0.$$

This is exact differential equation and we solve by the grouping method.

$$2dx - (\frac{y}{x^2}dx - \frac{1}{x}dy) + ydy = 0$$

Thus the general solution is

$$2x + \frac{y}{x} + \frac{y^2}{2} = c \blacksquare$$

Example 1.15 Find the general solution of $(4xy^2 + 6y)dx + (5x^2y + 8x)dy = 0$.

SOLUTION Since $M_y = 8xy + 6, N_x = 10xy + 8$, the given differential equation is not exact. Now $M_y - N_x = -2(xy + 1)$. Thus neither $(1/N)[M_y - N_x]$ nor $(1/M)[M_y - N_x]$ is a function of x only or y

only. Then we must find an integrating factor by the different method. M_y and N_x are polynomials. Then we let μ be $x^m y^n$. If $x^m y^n$ is an integrating factor, then

$$(4x^{m+1}y^{n+2} + 6x^m y^{n+1})dx + (5x^{m+2}y^{n+1} + 8x^{m+1}y^n)dy = 0$$

must be an exact differential equation.

$$\begin{aligned}\frac{\partial M}{\partial y} &= 4(n+2)x^{m+1}y^{n+1} + 6(n+1)x^m y^n \\ &= 5(m+2)x^{m+1}y^{n+1} + 8(m+1)x^m y^n = \frac{\partial N}{\partial x}\end{aligned}$$

Compare these equations to get

$$4(n+2) = 5(m+2),\ 6(n+1) = 8(m+1)$$

Solving the system of equations to get $n = 3, m = 2$. Thus

$$(4x^3y^5 + 6x^2y^4)dx + (5x^4y^4 + 8x^3y^3)dy = 0$$

is an exact differential equation, and the general solution is

$$x^4y^5 + 2x^3y^4 = c \blacksquare$$

1.5.1 Exercise

1. Find the general solution of the following differential equations.
(a) $(x - y^2)dx + 2xydy = 0$ (b) $(5xy + 4y^2 + 1)dx + (x^2 + 2xy)dy = 0$
(c) $(2xy^2 + y)dx + (2y^3 - x)dy = 0$ (d) $(2x + \tan y)dx + (x - x^2 \tan y)dy = 0$
2. Find the general solution of the following differential equations.
(a) $(2y^2 - xy)dx + (2x^2 - 3xy)dy = 0$ (b) $(y^4 + 2x^3y)dx - (x^4 + 2xy^3)dy = 0$

1.6 Linear Differential Equations

Linear differential equation

Given a 1st order linear differential equation in the normal form

$$y' + P(x)y = Q(x)$$

Write in the differential form

$$(P(x)y - Q(x))dx + dy = 0$$

Then

$$M_y = P(x),\ \ N_x = 0$$

and $(1/N)[M_y - N_x] = P(x)$ is a function of x only. Thus,

$$\mu = e^{\int P(x)dx}$$

Now multiplying $\mu(x) = e^{\int P(x)dx}$ to the normal form $y' + P(x)y = Q(x)$ to get

$$y'e^{\int P(x)dx} + P(x)ye^{\int P(x)dx} = Q(x)e^{\int P(x)dx}$$

The left-hand side is the derivative of $ye^{\int P(x)dx}$. Thus

$$ye^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx}dx + c.$$

Therefore the general solution is given by

$$y = e^{-\int P(x)dx}(\int Q(x)e^{\int P(x)dx}dx + c)$$

The 1st order linear differential equation can be solved by using the above formula. But it is much easier to use the integrating factor $e^{\int P(x)dx}$.

Example 1.16 Find the general solution of $y' + \frac{2}{x}y = x$.

SOLUTION We find the integrating factor.

$$\mu = e^{\int \frac{2}{x}dx} = e^{2\log x} = x^2.$$

Now multiplying the integrating factor to both sides.

$$x^2y' + 2xy = x^3.$$

Now

$$\frac{d(x^2y)}{dx} = x^3.$$

Integrating both sides to get

$$x^2y = \int x^3dx = \frac{x^4}{4} + c.$$

Thus the general solution is

$$y = \frac{x^2}{4} + \frac{c}{x^2} \blacksquare$$

RLS circuit

In a RLC circuit, the volatage drop by the circuit current i at the resistance $R(\Omega)$, at the inductance L(H), and at the capacitance C(F) is given by

$$\begin{aligned} 1.\ v_R &= Ri \\ 2.\ v_L &= L\frac{di}{dt} \\ 3.\ v_C &= \frac{1}{C}\int idt \end{aligned}$$

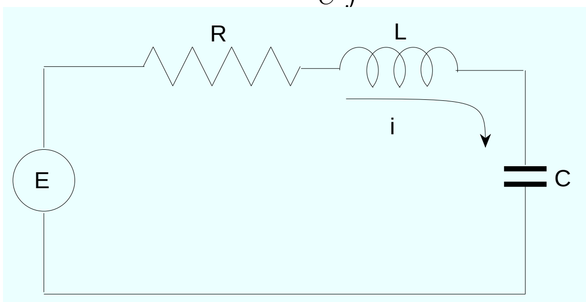

Figure 1.1: RLC AC circuit

Then by Kirchhoff'S 2nd law,

$$v_R + v_L + v_C = E.$$

or

$$L\frac{di}{dt} + R_i + \frac{1}{C}\int idt = E.$$

Example 1.17 Find the value of corrent i runs through the RL circuit, where R, L, E are constant, $i(0) = I_0$

SOLUTION By Kirchhoff's voltage law,

$$L\frac{di}{dt} + Ri = E.$$

Write in the normal form

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}.$$

Now the integrating factor μ is $e^{Rt/L}$. Then multiplying μ to both sides of the equation.

$$e^{Rt/L}(L\frac{di}{dt} + Ri) = \frac{E}{L}e^{Rt/L}.$$

Note that the left-hand side of the equation is the derivative of the integrating factor times the independent variable i. Thus

$$d(e^{Rt/L}i) = \frac{E}{L}e^{Rt/L}dt.$$

Thus the general solution is

$$i = \frac{E}{R} + ce^{-Rt/L}$$

Now using the initial value $i(0) = I_0$, we have $c = I_0 - (E/R)$. Therefore,

$$i = \frac{E}{R} + (I_0 - \frac{E}{R})e^{-Rt/L} \quad \blacksquare$$

1.6.1 Exercise

Find the general solution of the following differential equations.
(a) $y' \cos x - y \sin x + e^x = 0$ (b) $y' + 2xy = 2x$
(c) $xy' + y = x \sin x$ (d) $xy' + (1 + x)y = e^{-x} \sin 2x$
2. Solve the following initial problems.
(a) $y' + (\cos x)y = e^{-\sin x},\ y(0) = 2$
(b) $(x \log x)y' - y = \log x,\ y(e) = -1$
(c) $y' + y = f(x),\ y(0) = 0,\ f(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}$
(d) $y' + (\tan x)y = \cos^2 x,\ y(0) = -1$
3. Find $i(t)$, where $R = 10\Omega, E = 20V, L = \begin{cases} 5 - t, & 0 \le t \le 5 \\ 0, & 5 \le t \end{cases}$ closed cicuit with $i(0) = 0$.

1.7 Bernoulli, Riccati Equation

Bernoulli equation

If a differential equation is written in the form

$$y' + P(x)y = Q(x)y^{\alpha} \ (\alpha \neq 0, 1),$$

then it is called the **Bernoulli's equation**. we first get rid of y^{α} by multiplying $y^{-\alpha}$ to both sides of equation.

$$y^{-\alpha}y' + P(x)y^{(1-\alpha)} = Q(x)$$

Now let

$$u = y^{(1-\alpha)}$$

Then $u = y^{(1-\alpha)}$,$u' = (1-\alpha)y^{-\alpha}y'$ implies

$$\frac{1}{(1-\alpha)}u' + P(x)u = Q(x).$$

Note that this is a 1st order linear.

Example 1.18 Find the general solution of $y' + \frac{1}{x}y = -x^3y^2$.

SOLUTION This is a Bernoulli's equation. Then multiply y^{-2} to both sides of equation.

$$y^{-2}y' + \frac{1}{x}y^{-1} = -x^3$$

Now let $u = y^{-1}$. Then $u' = -y^{-2}y'$ and

$$-u' + \frac{1}{x}u = -x^3$$

This is a linear differential equation in u. So, rewrite this into the normal form.

$$u' - \frac{1}{x}u = x^3$$

Then $\mu = e^{-\int 1/x dx} = e^{-\log x} = 1/x$. Multiplying μ to both sides of the equation and noting the left-hand side becomes the derivative of the integrating factor times the independent variable u. Thus we have

$$d(\frac{u}{x}) = x^2 dt$$

Integrating both sides to obtain the general solution.

$$u = y^{-1} = \frac{x^4}{3} + cx \ \blacksquare$$

$\frac{df(y)}{dy}\frac{dy}{dx} + P(x)f(y) = Q(x)$

Given
$$\frac{df(y)}{dy}\frac{dy}{dx} + P(x)f(y) = Q(x),$$
let $u = f(y)$. Then since $\frac{du}{dx} = \frac{df(y)}{dy}\frac{dy}{dx}$,
$$\frac{du}{dx} + P(x)u = Q(x)$$

Example 1.19 Find the general solution of $\cos y \frac{dy}{dx} + \frac{1}{x}\sin y = 1$.

SOLUTION Let $u = \sin y$. Then $\frac{du}{dx} = \cos y \frac{dy}{dx}$. Thus the given differential equation is expressed in the form
$$\frac{du}{dx} + \frac{1}{x}u = 1.$$
This is a linear differential equation in u. Then the integrating factor $\mu = \int e^{\frac{1}{x}} dx = e^{\log x} = x$. Now multiply μ to both sides of the equation to get
$$d(xu) = xdx$$
Then $xu = \frac{1}{2}x^2 + c$. Solve for u to get
$$u = \frac{1}{2}x + cx^{-1}.$$
Since $u = \sin y$,
$$\sin y = \frac{1}{2}x + cx^{-1} \blacksquare$$

Riccati's equation

The differential equation expressed in the form
$$\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x)$$
is called the **Riccati's equation**. If a solution $f(x)$ is found, then we let $y = f(x) + \frac{1}{u}$. Now $\frac{dy}{dx} = f'(x) - \frac{1}{u^2}\frac{du}{dx}$. Thus
$$f'(x) - \frac{1}{u^2}\frac{du}{dx} = A(x)(f(x) + \frac{1}{u})^2 + B(x)(f(x) + \frac{1}{u}) + C(x)$$
This is a linear differential equation in u.

Example 1.20 Solve the following differential equation provided a solution $f(x) = 1$.
$$y' = (1-x)y^2 + (2x-1)y - x$$

SOLUTION This is a Riccati's equation. Since $f(x) = 1$ is a solution to the equation. Thus let

$$y = 1 + \frac{1}{u}.$$

Then $y' = -u'/u^2$. Substitute this into the differetial equation.

$$-\frac{u'}{u^2} = (1-x)(1+\frac{1}{u})^2 + (2x-1)(1+\frac{1}{u}) - x.$$

Now simplify the equation. Then we have

$$u' + u = 1 - x$$

Multiply $\mu = e^{\int dx} = e^x$ to both sides of the equation to get

$$d(e^x u) = e^x(1-x)dx.$$

Integrating both sides,

$$e^x u = e^x - xe^x + e^x + c = 2e^x - xe^x + c$$

Simplifying to get

$$u = 2 - x + ce^{-x}.$$

Since $y = 1 + 1/u$, we have

$$y = 1 + \frac{1}{2 - x + ce^{-x}} \blacksquare$$

1.7.1 Exercise

1. Find the general solution of the following differential equations.
(a) $xy' - y = -y^2$ (b) $xy' + y = y^2 \log x$
(c) $y' - y\cos x + y^2 \cos x = 0$
2. Solve the following differential equations.
(a) $yy' + y^2 + 4x(x+1) = 0$ (b) $(y+1)y' + x(y^2 + 2y) = x$
3. Solve the following differential equations.
(a) $x^2 y' = -x^2 y^2 - 4xy - 2$, provided a solution $f(x) = -2/x$
(b) $xy' = y - xy^2 + x^3$

1.8 Clairaut' equation

Clairaut's equation

In mathematics, a differential equation of the form

$$y = xy' + f(y')$$

is called the **Clairaut's equation**. To solve such a problem, we differentiate with respect to x, yielding

$$y' = y' + xy'' + f'(y')y''$$

Then $(x + f'(y'))y'' = 0$. Thus, either $y'' = 0$ or $x + f'(y') = 0$. In the former case, $y' = c$ for some constant. Substituting this into the Clairaut's equation, we have the family of straight line functions given by $y(x) = cx + f(c)$. The latter case $x + f'(y') = 0$. We let $\begin{cases} x = -f'(t) \\ y = -tf'(t) + f(t) \end{cases}$, where t is a parameter. Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-tf''(t)}{-f''(t)} = -t$$

Thus, $y = -tf'(t) + f(t) = \frac{dy}{dx}x + f(\frac{dy}{dx}) = xp + f(p)$. Then this is a solution to the Clairaut's equation. If $x = -f'(t)$ has a solution $t = h(x)$, then $y = xh(x) + f(h(x))$ and a singular solution to $y = xp + g(p)$.

d'Alembert's equation

A differntial equation of the form

$$y = xf(y') + g(y'),$$

where f and g are function of y', is called **D'Alembert equation**. To solve the D'Alembert's equation, we differentiate with respect to x.

$$y' = f'(y') + xf'(y')y'' + g'(y')y''$$

Rewriting

$$y' - f'(y') = (xf'(y') + g'(y'))y''$$

Now write this equation as

$$\frac{dx}{dy'} = \frac{f'(y')}{y' - f'(y')} + \frac{g'(y')}{y' - f'(y')}$$

Then this is a linear differential equation with dependent variable x and independent variable in y.

We use the following symbol for simplicity.

$$p = y' = \frac{dy}{dx}$$

Then the Clairaut's equation is expressed in the form

$$y = xp + f(p)$$

and the D'Alembert's equation is expressed in the form

$$y = xf(p) + g(p)$$

Example 1.21 Solve the following differential equation.

$$p^2 + (2x - y^2)p - 2xy^2 = 0$$

SOLUTION We solve for p. By factorization, we obtain $(p + 2x)(p - y^2) = 0$. This equation is satisfied by either $p + 2x = 0$ or $p - y^2 = 0$. Note that the general solution of $p + 2x = 0$ is $y + x^2 + C = 0$. Also, $p - y^2 = 0$ is separable differential equation. Thus we obtaine the general solution $xy + Cy + 1 = 0$. Form this the general solution is

$$(y + x^2 + C)(xy + Cy + 1) = 0$$

Example 1.22 Solve the defferential equation $xp^2 - 2yp = x$.

SOLUTION Solve for p. Then by the quadratic formula, we have

$$p = \frac{y \pm \sqrt{y^2 + x^2}}{x}$$

Since $p = \frac{dy}{dx}$, this is homogeneous. Let $y = vx$. Then $v + x\frac{dv}{dx} = v \pm \sqrt{1+v^2}$. Thus

$$\int \frac{dv}{\sqrt{1+v^2}} = \pm \int \frac{dx}{x}$$

$$\log|v + \sqrt{1+v^2}| = \pm \log|x| + C$$

$$\log|\frac{y}{x} + \sqrt{1+(\frac{y}{x})^2}| \mp \log|x| = C$$

Therefore,

$$y \mp \sqrt{x^2+y^2} = C$$

From this, we have the general solution

$$(y - \sqrt{x^2+y^2} - C)(y + \sqrt{x^2+y^2} - C) = 0$$

or

$$y = \frac{C^2 - x^2}{2C} = \frac{C}{2} - \frac{x^2}{2C}$$

Example 1.23 Solve the differntial equation $y = xp + p^{-1}$, where $y(1) = 2$

SOLUTION This is a Clairaut's equation. We first differentiate with respect to x.

$$p = p + x\frac{dp}{dx} - p^{-2}\frac{dp}{dx} = p + (x - p^{-2})\frac{dp}{dx}$$

Simplifying to get

$$(x - p^{-2})\frac{dp}{dx} = 0$$

From this, we have $\frac{dp}{dx} = 0$ or $x = p^{-2}$. For the first case, we have $p = c$ a some constant. Thus

$$y = xc + \frac{1}{c}$$

Since $y(1) = 1$, we have $2 = 1 + \frac{1}{c}$. Thus

$$y = x + 1$$

For the second case, put $x = p^{-2}$ into the original equation. Then

$$y = p^{-1} + p{-1} = 2p^{-1}$$

Rewrite this,

$$p = \frac{2}{y}$$

Thus

$$ydy = 2dx$$

$$\frac{y^2}{2} = 2x + c$$

Thus,

$$y^2 = 4x + c$$

Since $y(1) = 2$, we have $4 = 4 + c$. Therefore, the singular solution is

$$y^2 = 4x$$

Example 1.24 Solve the differential equation $yp^2 + 2xp = y$.

SOLUTION We first solve for x. Then $x = y(\frac{1}{2p} - \frac{p}{2})$. Now differentiate with respect to y, Then

$$\frac{1}{p} = p(\frac{1}{2p} - \frac{p}{2}) - y(\frac{1}{2p^2} + \frac{1}{2})\frac{dp}{dy}$$

Thus,

$$\frac{dp}{dy} = -\frac{\frac{1+p^2}{2p}}{y(\frac{1+p^2}{2p^2})} = -\frac{p}{y}$$

Simplifying to get $\frac{dp}{p} = -\frac{dy}{y}$. Then $\log|p| = -\log|y| + C$, $py = C$. From this, we have $yp^2 + 2xp = y$. Delete p and we have the general solution

$$C^2 + 2xC = y^2$$

Example 1.25 Solve the differential equation $x = 2p + \log|p|$.

SOLUTION We differentiate with respect to y.

$$\frac{1}{p} = 2\frac{dp}{dy} + \frac{1}{p}\frac{dp}{dy} = (2 + \frac{1}{p})\frac{dp}{dy}$$

Then

$$(2p + 1)dp = dy$$

Integrate with respect y

$$p^2 + p + C = y$$

Thus the general solution is given p as a parameter,

$$x = 2p + \log|p|,\ y = p^2 + p + C$$

1.8.1 Exercise

1. Solve the following differential equations, where $p = \frac{dy}{dx}$.
   - (a) $p^2 - y^2 = 0$
   - (b) $p^2 - 5p + 6 = 0$
   - (c) $y^2 + xyp - x^2p^2 = 0$
   - (d) $xy(1 - p^2) = (x^2 - y^2)p$
   - (e) $(x + 2y)p^3 + 3(x + y)p^2 + (2x + y)p = 0$
   - (f) $p^3 - 2p^2 + (2y - x^2 - \frac{y^2}{x^2})p - 2(2y - x^2 - \frac{y^2}{x^2}) = 0$
2. Solve the following differential equations, where $p = \frac{dy}{dx}$.
   - (a) $2x^3 = x^4p + p^3$
   - (b) $y + p^2 - 5xp + 5x^2 = 0$
   - (c) $x^4p^2 - xp - y = 0$
   - (d) $y = xp + \log p^2$
   - (e) $x^2 = 1 + p^2$

1.9 Numerical Solutions

We consider the 1st order differential equation $y' = f(x, y)$. We can think of y' as the slope of the tangent line at (x, y). Then consider the line starting at (x_0, y_0) with the slope $f(x_0, y_0)$. To draw this as graph, we first draw a curve $f(x, y) = c$. This curve is called **isocline**. Now we plot points on the curve, and at each point we draw a short pointed line with the slope $f(x, y)$. The collection of these pointed lines are called **direction field** of the differential equation $y' = f(x, y)$.

Up to this point, we try to find the general solution. Thus we ignore the singular solution. In this section, we treat the singular solution using the direction field. A general solutin with a singular solution is called the **complete solution**.

Example 1.26 Find the complete solution of $y' = \sqrt{1-y^2}$.

SOLUTION We first find the general solution.

$$\frac{dy}{\sqrt{1-y^2}} = \int dx$$
$$\sin^{-1} y = x + c$$
$$y = \sin(x+c)$$

We note that the derivative of right-hand side becomes negative. Thus the derivative of left-hand side y' becomes negative. But by the give differential equation, y' never becomes negative. Then we draw directional fiels using isocline lines .

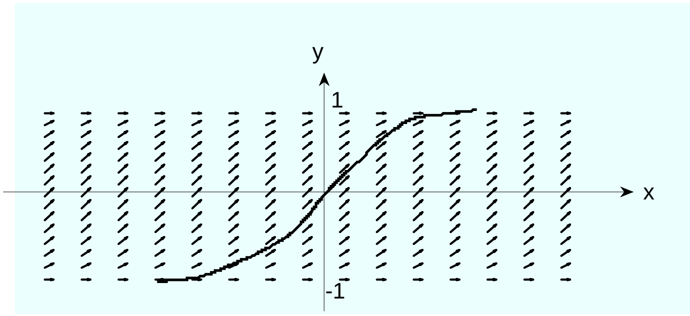

Figure 1.2: solutio curve

From this, $y \equiv 1$ and $y \equiv -1$ are solutions. Also, $y = 1$ と $y = -1$are connected with sin curve. Thus the complete solution is $y = 1$ or $y = -1$ or

$$y = \begin{cases} -1, & x + c \leq -\frac{\pi}{2} \\ \sin(x+c), & -\frac{\pi}{2} \leq x + c \leq -\frac{\pi}{2} \\ 1, & x + c \geq \frac{\pi}{2} \end{cases} \quad \blacksquare$$

We next consider the way to obtain numerical solution from the graph. The idea is to draw a line with the slope $f(x_0, y_0)$ at the starting point (x_0, y_0). Then the equation of line is given by

$$y = f(x_0, y_0)(x - x_0) + y_0.$$

Then take the next point as $x_1 = x_0 + h$ and $y_1 = f(x_0, y_0)h + y_0$. Now with the point (x_1, y_1) and the slope $f(x_1, y_1)$, we get

$$y = f(x_1, y_1)(x - x_1) + y_1.$$

Repeat this until we reach the given value of x.

Example 1.27 Given $y' = y + x^2, y(0) = 1$, find the value of $y(3)$ provided $h = 1$.

SOLUTION Since $(x_0, y_0) = (0, 1)$, we have $y = 1 + x$. Now since $(x_1, y_1) = (1, 2)$, we have $y = 3x - 1$. Finally, for $(x_2, y_2) = (2, 5)$, we have $y = 9x - 13$. Thus, $y(3) = 14$. Note that this differential equation is linear. Thus we can find the general solution. $y = 3e^x - x^2 - 2x - 2$. Then $y(3) = 3e^3 - 17 \doteq 43.3$ ■

As you noticed, the error by the approximation lines is large. To make the error small, we must use smaller h. The smaller the h, the more calsulation for computation. Then we use computer to calculate.

Euler's methos

The more useful technique is known as **Euler's method**.
The approximated solution of

$$y' = f(x, y), \ y(a) = b$$

can be obatinded by the following method.

$$\begin{array}{ll} x_0 = a & y_0 = b \\ x_1 = a + h & y_1 = y_0 + f(x_0, y_0)h \\ x_2 = a + 2h & y_2 = y_1 + f(x_1, y_1)h \\ \vdots & \vdots \\ x_n = a + nh & y_n = y_{n-1} + f(x_{n-1}, y_{n-1})h \end{array}$$

1.9.1 Exercise

1. Find the direction field and solution curve through the points $(1, 0), (0, 1), (1, 1)$.
(a) $y' = y$
(b) $y' - xy + 1 = 0$
2. Find the approximated solution of the following initial value problem by using Euler's method.
(a) For $y' = x + y, \ y(0) = 0$, find $y(3)$ provided $h = 1$
(b) For $y' = x^2, \ y(1) = 2$, find $y(4)$ provided $h = \frac{1}{2}$

Chapter 2

Linear Differential Equations

2.1 Solution of linear differential equation

nth-order linear differential equation

The differential equation of the form

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

is called the **nth-order linear differential equation**. $a_i(x)$ is called the **coefficient function**. $f(x)$ is called the **input function**.
If $f(x) \equiv 0$, then the differntial equation is called the **homogeneous equation**.

We denote the left-hand side as $L(y)$. Then the differential equation is expressed as

$$L(y) = f(x)$$

L is called the **differential operator** and has linearlity. That is for any solutions y_1, y_2 and constants c_1, c_2,

$$L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2)$$

Theorem 2.1 The solutions of homogeneous differential equations form a vector space.

We review the vector space.

Vector space

A sum of two vectors **A** and **B** is expressed as **A** + **B** and is equal to the diagonal of the parallelogram formed by **A** and **B**.

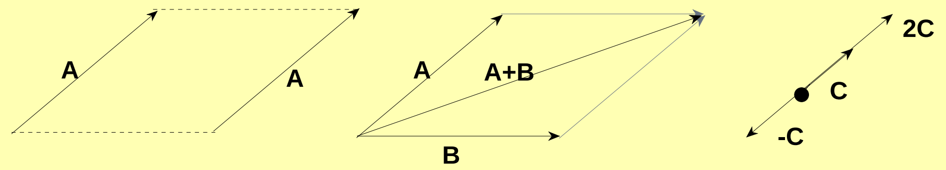

Figure 2.1: vector addition and scalar multiplication

1. A sum of two vetors is a vector (closure)
2. For any vectors **A** and **B**,**A**+**B** = **B**+**A** (commutative law)
3. For any vectors **A**,**B**,**C**, (**A**+**B**)+**C** = **A**+(**B**+**C**) (associative law)
4. Given any vector **A**, there exists a vector **0** satisying **A**+**0** = **A** (existence of zero)
5. Given any vector **A**, there exists a vector **B** satisfying **A**+**B** = **0** (existence of inverse)
6. A scalar multiplication of a vector is a vector
7. For any real numbers α and β, $\alpha(\beta\mathbf{A}) = (\alpha\beta)\mathbf{A}$ (associative law)
8. For any real numbers α and β, $(\alpha+\beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$ and for any vectors **A** and **B**, $\alpha(\mathbf{A}+\mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$ (distributive law)
9. $1\mathbf{A} = \mathbf{A}$; $0\mathbf{A} = \mathbf{0}$; $\alpha\mathbf{0} = \mathbf{0}$ (1 is multiplicative identity)

piecewise continuous function

Let $C(a,b)$ be the set of continuous functions on the interval (a,b). Let $PC(a,b)$ be the set of piecewise continuous functions on (a,b), [a]

[a] A function $f(t)$ is said to be piecewise continuous on the interval $[a,b]$ if the function satisies the following conditions: (i) $f(t)$ is continuous on the intervval $[a,b]$ except at finite number of points.
(ii) The left-hand limit and the right-hand limit exist at discontinuous point t_0 of $f(t)$.

$C(a,b) = \{f(x) : f(x) \text{ is continuous on } (a,b)\}$, $PC(a,b) = \{f(x) : f(x) \text{ is piecewise continuous on } (a,b)\}$.

For $f(x)$ and $g(x)$ in $C(a,b)$ or $PC(a,b)$, we define the addition on the scalar multiplication as follows:
1. $f+g$ is a function of x whose value is equal to $f(x)+g(x)$.
2. αf is a function of x whose value is equal to $\alpha f(x)$.

Example 2.1 Given $f(x) = x, g(x) = x^2$, find $f + g, \frac{1}{2}f, 2g$.

SOLUTION $(f + g)(x) = f(x) + g(x) = x + x^2$
$(\frac{1}{2}f)(x) = \frac{1}{2}f(x) = \frac{x}{2}$
$(2g)(x) = 2g(x) = 2x^2$ ■

vector space

Theorem 2.2 $C(a, b)$ with the operation defined above is a vector space. Then the function $f(x)$ belongs to $C(a, b)$ is called a vector.

Proof The solutions of a homogeneous differential equation are differentiable. Thus, the set of solutions are the subset of continuous functions. Then to show $C(a, b)$ is a vector space, it is enough to show the linear combination of the solutions y_1 and y_2 is a solution. Let $L(y_1) \equiv 0,\ L(y_2) \equiv 0, y_3 = c_1y_1 + c_2y_2$. Then

$$\begin{aligned} L(y_3) &= L(c_1y_1 + c_2y_2) \\ &= L(c_1y_1) + L(c_2y_2) \\ &= c_1L(y_1) + c_2L(y_2) \\ &= 0 \end{aligned}$$

Thus y_3 is again a solution. ■

The set of solutions of homogeneous equation becomes a vector space. So, we call this **solution space**.

Theorem 2.3 The basis of the solution space is a set of n independent solutions of homogeneous differential equation.

Wronskian determinant

The determinant of the following matrix is called **Wronskian determinant**. Let y_1, y_2 be the solutions of the differential equation. Then

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1, & y_2, & \cdots, & y_n \\ y_1', & y_2', & \cdots, & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}, & y_2^{(n-1)}, & \cdots, & y_n^{(n-1)} \end{vmatrix}.$$

Theorem 2.4 If $y_1, y_2, \ldots, y_n$ are the solutions of the homogeneous equation on the interval $[a, b]$, then $W(y_1, y_2, \ldots, y_n)$ is either 0 or never 0 0n the interval $[a, b]$.

Proof For $n = 2$ Since y_1 and y_2 are solutions of $L(y) = y'' + a_1 y' + a_0 y = 0$, we have

$$\begin{aligned} \frac{d}{dx}W(y_1, y_2) &= \begin{vmatrix} y_1' & y_2' \\ y_1' & y_2' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} \\ &= \begin{vmatrix} y_1 & y_2 \\ -a_1 y_1' - a_0 y_1 & -a_1 y_2' - a_0 y_2 \end{vmatrix} \\ &= -a_1 W(y_1, y_2). \end{aligned}$$

Thus

$$\frac{d}{dx}W(y_1, y_2) + a_1(x)W(y_1, y_2) = 0.$$

This is a linear differential equation. Thus the integrating factor $\mu(x) = \exp(\int_{x_0}^{x} a_1(t)dt)$. Multiplying μ to get

$$\frac{d}{dx}(\mu(x)W(y_1, y_2)) = 0$$

Integrating

$$W(y_1, y_2)(x) = c\exp(-\int_{x_0}^{x} a_1(t)dt) \ (c : \text{constant}).$$

Now let $x = x_0$. Then $c = W(y_1, y_2)(x_0)$ and

$$W(y_1, y_2)(x) = W(y_1, y_2)(x_0)\exp(-\int_{x_0}^{x} a_1(t)dt)$$

This becomes 0 or not is obvious ■

Theorem 2.5 If $y_1, y_2, \ldots, y_n$ are solutions of the homogeneous differential equation on $[a, b]$, then the following conditions are equivalent.
(1) $\{y_1, y_2, \ldots, y_n\}$ are linearly independent on the interval $[a, b]$.
(2) $W(y_1, \ldots, y_n) \neq 0, \ (x \in I)$

Proof For $n = 2$, let the linear combination of y_1 and y_2 be 0. Then

$$c_1 y_1 + c_2 y_2 = 0.$$

Differentiate with respect to x. Then

$$c_1 y_1' + c_2 y_2' = 0$$

Now using Cramer's rule to find c_1 and c_2.

$$c_1 = \frac{0}{W(y_1, y_2)}, \ c_2 = \frac{0}{W(y_1, y_2)}.$$

If $\{y_1, y_2\}$ are linearly independent, then $c_1 = c_2 = 0$. Thus, $W(y_1, y_2) \neq 0$. Conversely, if $W(y_1, y_2) \neq 0$, then $c_1 = c_2 = 0$. Thus,$\{y_1, y_2\}$ are linearly independent ■

Example 2.2 Find the general solution of $y^{(4)} = 0$.

SOLUTION The dimension of the solution space is 3. $y_1 = 1, y_2 = x, y_3 = x^2, y_4 = x^3$ are solutions of the differential equation. So, we need to show they are linearly independent.

$$W(1, x, x^2, x^3) = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 12.$$

Thus, Wronskian is not 0 and therefore linearly independent. The general solution is

$$y = c_1 + c_2x + c_3x^2 + c_4x^3 \blacksquare$$

Example 2.3 Suppose that $y = e^{mx}$ is a solution of $L(y) = y'' + 3y' + 2y = 0$. Then find the fundamental solution.

SOLUTION Since

$$\begin{aligned} L(e^{mx}) &= m^2e^{mx} + 3me^{mx} + 2e^{mx} \\ &= e^{mx}(m^2 + 3m + 2) \\ &= e^{mx}(m+1)(m+2), \end{aligned}$$

$m = -1, -2$ implies that $L(e^{mx}) \equiv 0$.

$$W(e^{-x}, e^{-2x}) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-2x} \neq 0$$

Thus, $\{e^{-x}, e^{-2x}\}$ are fundamental solutions. $\blacksquare$

Theorem 2.6 Suppose y_p is an particular solution of $n$the-order linear differential equation$L(y) = f(x)$ and y_c is the general solution of $L(y) = 0$. Then $y(x) = y_c(x) + y_p(x)$ is the general solution of $L(y) = f(x)$.

Proof By the assumption, $L(y_p) = f(x), L(y_c) = 0$. Now by the linearlity of L, we have

$$L(y_c + y_p) = L(y_c) + L(y_p) = f(x).$$

Thus $y_c + y_p$ is a solution of $L(y) = f(x)$. Since $y_c(x)$ contains n constants, $y_c(x) + y_p(x)$ is the general solution. $\blacksquare$

$y_c(x)$ is called the **complementary solution**. By this theorem, if an particular solution of $L(y) = f(x)$ is found, then to find the general solution, it is enough to find a complementary solution of $L(y) = 0$.

Example 2.4 Show $x - 1$ is an particular solution of $y'' + 3y' + 2y = 2x + 1$. Then find the general solution.

SOLUTION Since $y = x - 1, y' = 1, y'' = 0$, $y'' + 3y' + 2y = 2x + 1$. Also, the complementary solution of $L(y) = 0$ is given in example2.2. Thus

$$y_c(x) = c_1e^{-x} + c_2e^{-2x}.$$

Therefore, the general solution is

$$y(x) = \underbrace{c_1e^{-x} + c_2e^{-2x}}_{y_c} + \underbrace{x - 1}_{y_p} \blacksquare$$

If $f(x)$ is given as a sum of $f_1(x)$ and $f_2(x)$, then it is better to consider $L(y) = f_1(x)$ and $L(y) = f_2(x)$.

principal of superposition

Theorem 2.7 Suppose that y_{p_1} is a solution of $L(y) = f_1(x)$, and y_{p_2} is a solution of $L(y) = f_2(x)$. Then $y_{p_1} + y_{p_2}$ is a solution of $L(y) = f_1(x) + f_2(x)$.

Proof.

$$L(y_{p_1} + y_{p_2}) = L(y_{p_1}) + L(y_{p_2}) = f_1(x) + f_2(x)$$

For example, to find the general solution of the differential equation

$$y'' - 5y' + 2y = \sin x + x^2,$$

find (1) y_c of $y'' - 5y' + 2y = 0$
(2) y_{p_1} of $y'' - 5y' + 2y = \sin x$
(3) y_{p_2} of $y'' - 5y' + 2y = x^2$
and the linear combination

$$y = y_c + y_{p_1} + y_{p_2}$$

2.1.1 Exercise

1. The following differential equations have the solution of the form e^{mx}. Find the n independent solutions and the general solution. Finally,show the n solutions are linearly independent.
(a) $y''' + y'' - 10y' + 8y = 0$ (b) $y'' + 4y' + 3y = 0$
(c) $y'' - y' = 0$ (d) $y''' - 8y'' + 7y' = 0$
2. The following differential equations have the solution of the form either $\cos mx$ or $\sin mx$. Find the general solution.
(a) $y'' + 4y = 0$ (b) $y^{(4)} + 4y'' + 3y = 0$
3. The following differential equations have the solution of the form x^m. Find the general solution.
(a) $x^2y'' + xy' - 4y = 0$ (b) $x^2y'' - xy' - 3y = 0$

2.2 Reduction of order

Reduction of order

Let

$$L(y) = y'' + a_1(x)y' + a_0(x)y = f(x)$$

Suppose that a solution $y_1(x)$ of $L(y) = 0$ is known. Then substitute $y = u(x)y_1(x)$ into $L(y) = f(x)$.

$$(u''y_1 + 2u'y_1' + uy_1'') + a_1(x)(u'y_1 + uy_1') + a_0(x)(uy_1) = f(x).$$

or

$$y_1u'' + (2y_1' + a_1y_1)u' + (y_1'' + a_1y_1' + a_0y_1)u = f(x)$$

Since $L(y_1) = 0$, we note that the coefficient of u is 0. Now let $u' = w$. Then we have a 1st order linear differential equation in w.

$$y_1w' + (2y_1' + a_1y_1)w = f(x).$$

Example 2.5 Using the $y_1 = e^x$ is a solution of $y'' - y = 0$, find the general solution of

$$y'' - y = xe^x.$$

SOLUTION Let $y = uy_1 = ue^x$. Then $y' = u'e^x + ue^x, y'' = u''e^x + 2u'e^x + ue^x$. Substitute these into $y'' - y = xe^x$. Then

$$u''e^x + 2u'e^x = xe^x.$$

or

$$u'' + 2u' = x.$$

Now let $u' = w$. Then we have the following linear differential equation in w.

$$w' + 2w = x$$

$\mu = e^{\int 2dx} = e^{2x}$ and $d(e^{2x}w) = xe^{2x}dx$. Integrate this with respect to x to get

$$e^{2x}w = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c_1$$

Thus,

$$w = c_1e^{-2x} + \frac{x}{2} - \frac{1}{4}$$

Now integrate with respect to x

$$u = \int wdx = -\frac{c_1}{2}e^{-2x} + \frac{x^2}{4} - \frac{x}{4} + c_2$$

Since $y = ue^x$, the general solution is

$$y = -\frac{c_1}{2}e^{-x} + c_2e^x + (\frac{x^2}{4} - \frac{x}{4})e^x \blacksquare$$

Example 2.6 Using $L(x^2) \equiv 0$, Reduce the order of the following differential equation

$$L(y) = y''' + 2x^2y'' - 3xy' + 2y = 0$$

SOLUTION Let $y = ux^2$. Then $y' = u'x^2 + 2ux$, $y'' = u''x^2 + 4u'x + 2u$, $y''' = u'''x^2 + 6u''x + 6u'$. Thus

$$L(y) = L(ux^2) = x^2u''' + (2x^4 + 6x)u'' + (5x^3 + 6)u' = 0$$

Now let $u' = w$. Then

$$x^2w'' + (2x^4 + 6x)w' + (5x^3 + 6)w = 0 \blacksquare$$

2.2.1 Exercise

1. Solve the following differential equation by the reduction of order.
 (*a*) $y'' - 3y' + 2y = 0,\ y_1 = e^{2x}$ (*b*) $x^2y'' - 3xy' + 4y = 0,\ y_1 = x^2$
 (*c*) $y'' - y = e^{-x},\ y_1 = e^{-x}$ (*d*) $y'' + y = \sec x,\ y_1 = \cos x$
2. Suppose $y_1(x)$ is a solution of $y'' + a_1(x)y' + a_0(x)y = 0$. Then an another solution $y_2(x)$ is given by

$$y_2(x) = y_1(x)\int \frac{e^{-\int a_1(x)dx}}{y_1^2}dx.$$

Also, show that $\{y_1$ and $y_2\}$ are linearly independent.

2.3 Higher order homogeneous linear differential equation

Characteristic equation

Note that $y = ce^{mx}$ is a complete solution of $y' = my$. Then we use $y = e^{mx}$ as a candidate for the solution of the following nth-order linear differential equation.

$$L(y) = a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Then since

$$L(e^{mx}) = a_n m^n e^{mx} + a_{n-1}m^{n-1}e^{mx} + \cdots + a_1 m e^{mx} + a_0 e^{mx} = 0$$

e^{mx} is a solution of $L(y) = 0$ if and only if

$$P(m) = a_n m^n + a_{n-1}m^{n-1} + \cdots + a_1 m + a_0 = 0$$

The polynomial $P(m)$ is called the **characteristic polynomial**, and $P(m) = 0$ is called the **characteristic equation**. This way, we only need to solve the polynomial equation $P(m) = 0$ instead of solving $L(y) =$.

Example 2.7 Solve the differential equation $y'' + 3y' + 2y = 0$.

SOLUTION The roots of the characteristic equation $m^2 + 3m + 2 = 0$ are $m = -1, m = -2$. Then e^{-x} and e^{-2x} are solutions and by example 2.2, these solutions are linearly independent. Thus, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} \blacksquare$$

Distince real roots

Theorem 2.8 Suppose that the roots of the characteristic polynomial $P(m) = 0$ of the nth -order linear differential equation are distinct real roots $m = r_1, r_2, \ldots, r_n$. Then the set of solutions $\{e^{r_1 x}, e^{r_2 x}, \cdots, e^{r_n x}\}$ is a basis for the solution space.

Proof. We show $\{e^{r_1 x}, e^{r_2 x}, \cdots, e^{r_n x}\}$ are linearly independent by using Wronskian. Then

$$W = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} & \cdots, & e^{r_n x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} & \cdots & r_n e^{r_n x} \\ r_1^2 e^{r_1 x} & r_2^2 e^{r_2 x} & \cdots & r_n^2 e^{r_n x} \\ \vdots & \vdots & \vdots & \vdots \\ r_1^{n-1} e^{r_1 x} & r_2^{n-1} e^{r_2 x} & \cdots & r_n^{n-1} e^{r_n x} \end{vmatrix} = e^{r_1 x} e^{r_2 x} \cdots e^{r_n x} \begin{vmatrix} 1 & 1 & \cdots, & 1 \\ r_1 & r_2 & \cdots & r_n \\ r_1^2 & r_2^2 & \cdots & r_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^2 \end{vmatrix}$$

Now this determinant is the Vandermonde determinant. Thus we have $W = \Pi_{1 \le i,j \le n-1}(r_j - r_i)$. Since r_i are distinct real roots, $W \neq 0$.

Example 2.8 Suppose the roots of the characteristic equation $P(m) = 0$ of the 4th-order homogeneous linear differential equation $L(y) = 0$ are $m = 1, 2, 3, 4$. Then find the general solution.

SOLUTION By the theorem above, $\{e^x, e^{2x}, e^{3x}, e^{4x}\}$ is the basis of the solution space. Thus, the general solution is the linear combination of $\{e^x, e^{2x}, e^{3x}, e^{4x}\}$. Therefore,

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x} + c_4e^{4x} \blacksquare$$

Example 2.9 Solve the differential equation $y'' - 2y' + y = 0$.

SOLUTION The roots of the characteristic equation $m^2 - 2m + 1 = 0$ are $m = 1, 1$. Then $y_1 = e^x$ is a solution. Since this differential equation is the 2nd-order, we must have another linearly independent solution. By exercise 2.2.1, y_2 can be obtained by

$$\begin{aligned} y_2 &= e^x \int \frac{e^{-\int -2dx}}{e^{2x}} dx \\ &= e^x \int \frac{e^{2x}}{e^{2x}} dx = xe^x \end{aligned}$$

Thus the general solution is

$$y = c_1y_1 + c_2y_2 = c_1e^x + c_2xe^x \blacksquare$$

Repeated roots

Theorem 2.9 Suppose the roots of the characteristic equation $P(m) = 0$ are k-fold multiple roots $m = r$. Then x^ne^{rx}, $(n = 0, 1, \ldots, k-1)$ are the solutions of the differential equation $L(y) = 0$. Furthermore, the set of solutions $\{x^ne^{rx} : n = 0, 1, \ldots, k-1)\}$ is a basis of the solution space.

Proof Denote $D = \frac{d}{dx}$. Then we can express

$$L(y) = a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y$$

as

$$L(D)y = (a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y$$

Since $L(e^{mx}) = e^{mx}P(m)$, we have $L(D)e^{mx} = e^{mx}P(m)$. Now evaluate $D^i(x^ne^{mx})$. Then

$$D^i(x^ne^{mx}) = e^{mx}(D+m)^ix^n$$

Then

$$L(D)x^ne^{mx} = e^{mx}P(D+m)x^n.$$

Thus for $m = r$ is k-fold multiple roots, $L(D) = (D-r)^k$ and

$$L(D)x^ne^{rx} = e^{rx}D^kx^n = 0 \ (n = 1, 2, \ldots, k-1).$$

Next we show x^ne^{rx} $(n = 0, 1, \ldots, k-1)$ are linearly independent.

$$c_1e^{rx} + c_2xe^{rx} + \cdots + c_{k-1}x^{k-1}e^{rx} = 0$$

Since $r^{rx} \neq 0$, we have

$$c_1 + c_2x + c_3x^2 + \cdots + c_{k-1}x^{k-1} = 0.$$

Thus, $c_1 = c_2 = c_3 = \cdots = c_{k-1} = 0$. $\blacksquare$

Example 2.10 If the roots of the characteristic equation $P(m) = 0$ of the 4th-order linear differential equation $L(y) = 0$ are $m = -3, -3, -3, -3$, then find the general solution.

SOLUTION By the theorem 2.9, $e^{-3x}, xe^{-3x}, x^2e^{-3x}, x^3e^{-3x}$ are linearly independent solutions of $L(y) = 0$. Thus the general solution is

$$y = c_1e^{-3x} + c_2xe^{-3x} + c_3x^2e^{-3x} + c_4x^3e^{-3x} \blacksquare$$

Complex roots

Let the coefficients of $L(y)$ be real. If $a + bi$ is the root of the characteristic equation $P(m) = 0$, then the conjugate $a - bi$ is also a root. Thus,

$$y_1 = e^{(a+bi)x}, y_2 = e^{(a-bi)x}$$

are solutions of $L(y) = 0$. Now by the **Euler's formula**, $e^{i\theta} = \cos\theta + i\sin\theta$, and the linearlity of solutions

$$\begin{aligned} y_3 &= \frac{y_1 + y_2}{2} = e^{ax}\cos bx, \\ y_4 &= \frac{y_1 - y_2}{2i} = e^{ax}\sin bx \end{aligned}$$

are solutions of $L(y) = 0$. It is not hard to show $\{y_3$ and $y_4\}$ are linearly independent. Thus, the basis of the solutions corresponding to k-fold multiple roots of complex numbers is

$$\{e^{ax}\cos bx, e^{ax}\sin bx, xe^{ax}\cos bx, x^{ax}\sin bx, \ldots, x^{k-1}e^{ax}\cos bx, x^{k-1}e^{ax}\sin bx\}.$$

Example 2.11 Solve the differential equation $y'' + 2y' + 4y = 0$.

SOLUTION The roots of the characteristic equation $m^2 + 2m + 4 = 0$ are $m = -1 \pm i\sqrt{3}$. Then

$$\{e^{-x}\cos\sqrt{3}x, e^{-x}\sin\sqrt{3}x\}$$

is the fundamental solution. Thus the general solution is

$$y = c_1e^{-x}\cos\sqrt{3}x + c_2e^{-x}\sin\sqrt{3}x \blacksquare$$

The mass of the object is m, the spring constant is k, The force loss due to friction of dashpot is proportional to the speed of the object dy/dt. Then the forces acting on the object is given by

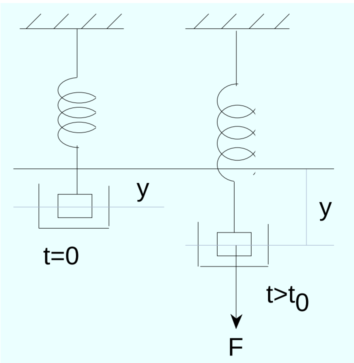

Figure 2.2: vibration of spring

1. $F_1 \quad = mg$ the gravitational force
2. $F_2 \quad = -ky$ the restoring force by the spring
3. $F_3 \quad = -a\frac{dy}{dt}$ $(a > 0)$ force due to friction
4. $F_4 \quad = F(t)$ the external force

Now using the Newton's 2nd law, we have

$$m\frac{d^2y}{dt^2} = mg - ky - a\frac{dy}{dt} + F(t).$$

or

$$m\frac{d^2y}{dt^2} + a\frac{dy}{dt} + ky = F(t) + mg$$

Now note that when the external force $F(t) = 0$, $ky_0 = mg$. Then for $y + y_0$, we have

$$m\frac{d^2y}{dt^2} + a\frac{dy}{dt} + k(y + y_0) = F(t) + mg$$

Therefore, when the system is at equilibrium with gravity, we can ignore the gravity and

$$m\frac{d^2y}{dt^2} + a\frac{dy}{dt} + ky = F(t)$$

Example 2.12 Suppose that the massless spring is hanging from the ceiling. When you attached a small object weighing $30g$ to its lower end, it stretched $2cm$. From this position you have stretched the spring $5cm$ and release. Find the motion of the spring.

SOLUTION By Hooke's law, we have the spring constant $k = 30/2 = 15$. Then by the Newton's 2nd law, we have

$$30\frac{d^2y}{dt^2} + 15y = 0$$

This is the 2nd-order linear differential equation. The characteristic equation is

$$m^2 + \frac{1}{2} = 0.$$

Solve this to get $m = \pm\frac{i}{\sqrt{2}}$. Thus,

$$y = c_1 \cos\frac{t}{\sqrt{2}} + c_2 \sin\frac{t}{\sqrt{2}}.$$

Now using the initial conditions $y(0) = 5,\ y'(0) = 0$, we have

$$y = 5\cos\frac{t}{\sqrt{2}} \quad \blacksquare$$

2.3.1 Exercise

1. Find the fundamental solutions of the following differential equations.

(a) $y'' + 9y = 0$ $\quad$ (b) $y''' + y = 0$

(c) $y''' - 3y' - 2y = 0$ $\quad$ (d) $y^{(5)} + 18y''' + 81y' = 0$

2. Find the general solutions.

(a) The roots of the characteristic equation of the 4th-order linear homogeneous differential equation are $m = -2, 1, 1, 1$

(b) The roots of the characteristic equation of the 6th-order linear homogeneous differential equation are $m = 0, 0, -1 \pm 2i, -1 \pm 2i$

3. Solve the following initial value problems.
(a) $y'' - y = 0,\ y(0) = 1, y'(0) = 1$
(b) $y'' - 6y' + 9y = 0,\ y(0) = 1, y'(0) = 2$
(c) $y''' + 7y'' + 19y' + 13y = 0,\ y(0) = 0, y'(0) = 2, y''(0) = -12$
(d) $y^{(4)} + 2y''' + 10y'' = 0,\ y(0) = 5, y'(0) = -3, y''(0) = 0, y'''(0) = 0$

2.4 Method of undetermined coefficients

How to find the particular solutions of $L(y) = f(x)$

Suppose that $f(x)$ is the solution of the homogeneous linear differential equation. Let $D = \frac{d}{dx}$. Then

$$L(y) = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

can be expressed as

$$L(D)y = (a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0)y = f(x)$$

Now since $f(x)$ is the solution of the homogeneous linear differential equation, there exists a polynomial $H(D)$ satisfying $H(D)f(x) = 0$. Let y_p be the particular solution of $L(D)y = f(x)$. Then

$$H(D)L(D)y_p = H(D)f(x) = 0$$

Thus to find y_p, all we need to find is solutions of $H(D)L(D)y = 0$ satisfying $L(D)y = f(x)$. This method is called the **method of undetermind coefficient**.

$$\begin{aligned} f(x) &= x^m \Longrightarrow H(D)f(x) = D^{m+1}x^m = 0 \\ f(x) &= e^{ax} \Longrightarrow H(D)f(x) = (D-a)e^{ax} = 0 \\ f(x) &= e^{ax}\cos bx \Longrightarrow H(D)f(x) = (D^2 - 2aD + a^2 + b^2)e^{ax}\cos bx = 0 \\ f(x) &= e^{ax}\sin bx \Longrightarrow H(D)f(x) = (D^2 - 2aD + a^2 + b^2)e^{ax}\sin bx = 0 \end{aligned}$$

Example 2.13 Solve the differential equation $L(y) = y'' - y' - 2y = 2e^{3x}$.

SOLUTION The characteristic equation of $L(y) = 0$ is $m^2 - m - 2 = 0$. Then the roots of characteristic equation are $m = -1, 2$. Then the complementary solution y_c is

$$y_c = c_1 e^{-x} + c_2 e^{2x}$$

Now we find the particular solution y_p using the method of undetermined coefficients. Let $D = \frac{d}{dx}$. Then since $f(x) = 2e^{3x}$, let $H(D) = D - 3$. Then $H(D)2e^{3x} = 2(D-3)e^{3x} = 0$. Thus y_p is a solution of

$$(H(D)L(D))y = 2(D-3)(D^2 - D - 2)y = 0$$

The characteristic equation of this is $2(m-3)(m^2 - m - 2) = 0$. Thus the fundamental solutions are e^{3x}, e^{-x}, e^{2x}. Since e^{-x}, e^{2x} satisfy $L(D)y = 0$. Thus we set y_p as

$$y_p = Ae^{3x}$$

Now put this back into $L(D)y = 2e^{3x}$. Then

$$L(D)Ae^{3x} = (Ae^{3x})'' - (Ae^{3x})' - 2(Ae^{3x}) = 4Ae^{3x} = 2e^{3x}$$

From this, we have $A = \frac{1}{2}$. Thus $y_p = \frac{1}{2}e^{3x}$ and the general solution is

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{2}e^{3x} \blacksquare$$

Example 2.14 Solve the differntial equation $L(y) = y'' - y' - 2y = e^{-x}$.

SOLUTION The characteristic equation of $L(y) = 0$ is given by $m^2 - m - 2 = 0$. Thus we have $m = -1, 2$. Then the complementary solution y_c is

$$y_c = c_1 e^{-x} + c_2 e^{2x}$$

We next find y_p using the method of undetermined coefficients. Since $H(D)e^{-x} = (D+1)e^{-x} = 0$, y_p is a solution of

$$H(D)L(D)y = (D+1)(D^2 - D - 2)y = (D+1)^2(D-2)y = 0$$

Then the fundamental solutions are e^{-x}, xe^{-x}, e^{2x}. But e^{-x}, e^{2x} are solution of $L(y) = 0$. Thus we set

$$y_p = Axe^{-x}$$

Put this back into $L(D)y = e^{-x}$. Then

$$L(D)Axe^{-x} = -3Ae^{-x} = e^{-x}$$

Thus,

$$y_p = -\frac{1}{3}xe^{-x},$$

and the general solution is

$$y = c_1 e^{-x} + c_2 e^{2x} - \frac{1}{3}xe^{-x} \blacksquare$$

Example 2.15 Solve the differential equation $L(y) = y'' + 2y' = e^x - x^2$.

SOLUTION The characteristic equation of $L(y) = 0$ is $m^2 + 2m = 0$. Then we have $m = 0, -2$. Thus, the complementary solution y_c is

$$y_c = c_1 + c_2 e^{-2x}$$

Now to find y_p, we find the particular solution y_{p_1} of $L(D)y = e^x$ and y_{p_2} of $L(D)y = x^2$. Then y_P is given by $y_p = y_{p_1} + y_{p_2}$.

The particular solution y_{p_1} of $L(D)y = e^x$ can be found by setting $y_{p_1} = Ae^x$. Also the particular solution y_{p_2} of $L(D)y = x^2$can be found by setting

$$H(D)L(D)y = (D^3 D(D+2))y = 0$$

Then $1, x, x^2, x^3, e^{-2x}$ are the fundamental solutions.But 1 and e^{-2x} are complementary solutions. Thus set

$$y_{p_2} = Bx + Cx^2 + Dx^3$$

Now let $y_p = y_{p_1} + y_{p_2}$ and substitute into $L(D)y = e^x - x^2$. Then $L(D)(Ae^x + Bx + Cx^2 + Dx^3) = 3Ae^x + 2C + 2B + (6D + 4C)x = e^x - x^2$. Thus $A = \frac{1}{3}, B = -\frac{1}{4}, C = \frac{1}{4}, D = -\frac{1}{6}$. Therefore, the general solution is

$$y = c_1 + c_2 e^{-2x} + \frac{1}{3}e^x - \frac{1}{4}x + \frac{1}{4}x^2 - \frac{1}{6}x^3 \blacksquare$$

2.4.1 Exercise

1. Solve the following differential equation using the method of undetermined coefficient.

(a) $y'' - 4y' + 4y = e^x$ (b) $y'' - 4y' + 4y = e^{2x}$

(c) $y'' + 4y = \sin 2x$ (d) $y'' - 3y' + 2y = e^x + e^{2x} + e^{-x}$

(e) $y''' - 3y' - 2y = e^{2x}\sin 2x$ (f) $y^{(4)} + 2y'' + y = x^2 e^x$

2.5 Variation of parameters

variation of parameters for the 2nd-order linear differential equation

Consider

$$L(y) = y'' + a_1 y' + a_0 y = f(x)$$

y_1 and y_2 be the fundamental solution of $L(y) = 0$. Let the particular solution be as follows

$$y_p = u_1 y_1 + u_2 y_2$$

Here $u_1(x)$ and $u_2(x)$ are undeterminded functions. To find $u_1(x)$ and $u_2(x)$, we need two conditions. The first condition is that $y_p = u_1y_1 + u_2y_2$ is a solution of $L(y) = f(x)$. The second condition is to make the calculation simple, that is,

$$u_1'y_1 + u_2'y_2 = 0$$

To satisfy the 1st condition, we find y', y'' and substitute into $L(y) = f(x)$. Then

$$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

Now use the second condition $u_1'y_1 + u_2'y_2 = 0$,

$$y' = u_1y_1' + u_2y_2'$$

Thus

$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

Substitute these into $L(y) = y'' + a_1y' + a_0y = f(x)$. Then

$$(u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + a_1(u_1y_1' + u_2y_2') + a_0(u_1y_1 + u_2y_2) = f(x)$$

Simplifying

$$(u_1'y_1' + u_2'y_2') + u_1(y_1'' + a_1y_1' + a_0y_1) + u_2(y_2'' + a_1y_2' + a_0y_2) = f(x).$$

Now y_1 and y_2 are solutons of $L(y) = 0$. Thus the coefficients of u_1 and u_2 are 0. Therefore, u_1' and u_1' are solutions of the following system of equation.

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= f(x) \end{aligned}$$

From this, solve for u_1' and u_2' and then find u_1 and u_2.

To solve the above system, we use the Cramer's rule. Then

$$u_1' = \frac{-y_2 f(x)}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \; u_2' = \frac{y_1 f(x)}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

Note that the denominator of the above system is Wronskian of y_1 and y_2. Since y_1 and y_2 are linearly independent. Thus by the theorem 2.5, the Wronskian is never 0. Thus we can find u_1' and u_2'. By integrating with respect to x, we can find u_1 and u_2. Thus we can find $y_p = u_1y_1 + u_2y_2$.

Example 2.16 Solve the differentiation equation $L(y) = y'' + y = \sec x$.

SOLUTION The characteristic equation of $L(y) = 0$ is $m^2+1 = 0$ and thus the roots are $m = \pm i$. Then the complementary solution is

$$y_c = c_1 \cos x + c_2 \sin x$$

Next we find the particular solution y_p. Since $f(x) = \sec x$ is not a solution of a homogeneous linear differential equation, we can not use the method of undetermined coefficients. So, we let

$$y_p = u_1 \cos x + u_2 \sin x$$

Then

$$u_1' = \frac{-\sin x \sec x}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}},\ u_2' = \frac{\cos x \sec x}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}}.$$

Thus,

$$u_1' = -\sin x \frac{1}{\cos x} = -\tan x,\ u_2' = \cos x \frac{1}{\cos x} = 1$$

Integrate with respect to x, we have

$$u_1 = \log|\cos x|,\ u_2 = x$$

Thus the general solution is

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x + (\log|\cos x|)\cos x + x \sin x \ \blacksquare$$

Variation of parameters for nth-order linear differential equation

Suppose that

$$L(y) = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

Let $y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ be the solution of $L(y) = 0$. Now replace the constants $c_1, c_2, \ldots, c_n$ by the variables $u_1, u_2, \ldots, u_n$. Then

$$y_p = u_1y_1 + u_2y_2 + \cdots + u_ny_n$$

Now $u_1', u_2', \ldots, u_n'$ satisfy the following system.

$$\begin{cases} u_1'y_1 + \cdots + u_n'y_n & = 0 \\ u_1'y_1' + \cdots + u_n'y_n' & = 0 \\ \vdots & \vdots \\ u_1'y_1^{(n-1)} + \cdots + u_n'y_n^{(n-1)} & = f(x) \end{cases}$$

Example 2.17 Solve the following differential equation

$$y''' - y' = \frac{1}{1+e^x}$$

SOLUTION The characteristic equation of $y''' - y' = 0$ is $m^3 - m = 0$. Then roots are $m = 0, \pm 1$. Thus the complementary solution is

$$y_c = c_1 + c_2 e^x + c_3 e^{-x}$$

Since $f(x) = 1/(1+e^x)$, we use the variation of parameter to find the particular solution. Let

$$y_p = u_1 + u_2 e^x + u_3 e^{-x}$$

Then

$$\begin{cases} u_1' + u_2' e^x + u_3' e^{-x} & = 0 \\ u_2' e^x - u_3' e^{-x} & = 0 \\ u_2' e^x + u_3' e^{-x} & = \frac{1}{1+e^x} \end{cases}$$

By the Cramer's rule, we have

$$\begin{aligned} u_1' &= \frac{\begin{vmatrix} 0 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ \frac{1}{1+e^x} & e^x & e^{-x} \end{vmatrix}}{\begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix}} = \frac{1}{2}\frac{-2}{1+e^x} = \frac{-1}{1+e^x}, \\ u_2' &= \frac{\begin{vmatrix} 1 & 0 & e^{-x} \\ 0 & 0 & -e^{-x} \\ 0 & \frac{1}{1+e^x} & e^{-x} \end{vmatrix}}{2} = \frac{\frac{1}{2}e^{-x}}{1+e^x}, \\ u_3' &= \frac{\begin{vmatrix} 1 & e^x & 0 \\ 0 & e^x & 0 \\ 0 & e^x & \frac{1}{1+e^x} \end{vmatrix}}{2} = \frac{\frac{1}{2}e^x}{1+e^x}. \end{aligned}$$

Thus

$$\begin{aligned} u_1 &= \int \frac{-dx}{1+e^x} = -\int \frac{e^{-x}dx}{e^{-x}+1} = \log(e^{-x}+1), \\ u_2 &= \int \frac{\frac{1}{2}e^{-x}}{1+e^x}dx = \frac{1}{2}\int (e^{-x} - \frac{1}{1+e^x})dx = -\frac{1}{2}e^{-x} + \frac{1}{2}\log(e^{-x}+1), \\ u_3 &= \frac{1}{2}\int \frac{e^x}{1+e^x} = \frac{1}{2}\log(1+e^x) \end{aligned}$$

Put these back to y_p and we have the general solution

$$y = y_c + y_p = c_1 + c_2 e^x + c_3 e^{-x} - \frac{1}{2} + (1 + \frac{1}{2}e^x)\log(e^{-x}+1) + \frac{1}{2}e^{-x}\log(1+e^x) \ \blacksquare$$

2.5.1 Exercise

1. Solve the following differential equation using the variation of parameter.
 (a) $y'' + 2y' + y = \frac{e^{-x}}{x}$ (b) $y'' + 4y = \tan 2x$
 (c) $y'' - 4y' + 4y = \frac{e^{2x}}{x}$ (d) $y'' - 3y' + 2y = \frac{1}{1+e^{-x}}$
 (e) $y''' + y' = \tan x$
2. Using the variation of parameter, show the general solution of $y'' + y = f(x)$ is given by

$$y = c_1 \cos x + c_2 \sin x + \int_a^x f(t)\sin(x-t)dt$$

2.6 Cauchy-Euler Equation

Cauchy-Euler equation

Consider 2nd-order linear differential equation with variable coefficients.

$$a_2x^2y'' + a_1xy' + a_0y = f(x)\ (x > 0)$$

Let $x = e^t$. Then for $x > 0$, we have $t = \log x$ and

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt},$$

Thus

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}(\frac{1}{x})\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}(\frac{dy}{dt}) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}(\frac{d^2y}{dt^2}\frac{dt}{dx}) \\ &= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}(\frac{d^2y}{dt^2}\frac{1}{x}) = \frac{1}{x^2}(\frac{d^2y}{dt^2} - \frac{dy}{dt}).\end{aligned}$$

From this,

$$x\frac{dy}{dx} = \frac{dy}{dt},\ x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Put these back to the original equation. We have the following linear differential equation with the constant coefficients.

$$a_2(\frac{d^2y}{dt^2} - \frac{dy}{dt}) + a_1\frac{dy}{dt} + a_0y = f(e^t).$$

The characteristic equation of the Cauchy-Euler equation is given by

$$a_2m^2 + (a_1 - a_2)m + a_0 = 0$$

and is called the **indicial equation**.

There is a simple way to find the indicial equation. Substitute $y = x^m$ into the following differential equation.

$$a_2x^2y'' + a_1xy' + a_0y = 0(x > 0)$$

Then we have

$$a_2x^2m(m-1)x^{m-2} + a_1xmx^{m-1} + a_0x^m = (a_2m^2 + (a_1 - a_2)m + a_0)x^m = 0$$

Example 2.18 Solve the following differetial equation

$$x^2y'' - 2xy' + 2y = 0\ (x > 0)$$

SOLUTION Let $x = e^t$ and $y = x^m$. Then the indicial equation is $m(m-1) - 2m + 2 = 0$. Thus $m = 1, 2$. Furtheremore, the indicial equation is the characteristic equation of

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0$$

The general solution is

$$y = c_1e^t + c_2e^{2t} = c_1x + c_2x^2\ \blacksquare$$

Example 2.19 Solve the following differential equation

$$x^2y'' - 2xy' + 2y = x^3 \ (x > 0)$$

SOLUTION The example above, we found the complementary solution is $y_c = c_1e^t + c_2e^{2t}$. Now since $f(t) = e^{3t}$, we find the particular solution by the method of undetermined coefficients. Substitute $y_p = Ae^{3t}$ into

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$$

Then

$$2Ae^{3t} = e^{3t}$$

Thus, $A = \frac{1}{2}$ and the general solution is

$$y = c_1e^t + c_2e^{2t} + \frac{1}{2}e^{3t}$$

Now since $x = e^t$, we have

$$y = c_1x + c_2x^2 + \frac{x^3}{2} \ \blacksquare$$

2.6.1 Exercise

1. For $x > 0$, solve the following Cauchy-Euler equation.
(*a*) $x^2y'' + 4xy' + 2y = 0$ (*b*) $x^2y'' + xy' + 9y = 0$
(*c*) $x^2y'' - xy' - 3y = x^2 \log x$ (*d*) $x^3y''' + x^2y'' - 2xy' + 2y = x^3$
(*e*) $x^2y''' + 5xy'' + 4y' = 0$

2.7 Differential Operator

Let $D = \frac{d}{dx}$. Then we can write

$$L(y) = a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = f(x)$$

as

$$L(D)y = (a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y = f(x)$$

Then the operator gives the function y for $L(D)y = f(x)$ is called the **Inverse operator** and is expressed by

$$y = \frac{1}{L(D)}f(x) = (L(D))^{-1}f(x)$$

Thus $(L(D))^{-1}f(x)$ applies to the function of $f(x)$ such that

$$L(D)(L(D)^{-1}f(x)) = f(x)$$

Basic rule

Theorem 2.10 (linearlity) Let α, β be sonstants, $f(x), g(x)$ be functions. Then

$$L(D)(\alpha f(x) + \beta g(x)) = \alpha L(D)f(c) + \beta L(D)g(x)$$

$$L(D)^{-1}(\alpha f(x) + \beta g(x)) = \alpha L(D)^{-1}f(x) + \beta L(D)^{-1}g(x)$$

Proof The first one comes from the definition of $L(D)$. The second one comes from the following.

$$L(D)(\alpha L(D)^{-1}f(x) + \beta L(D)^{-1}g(x)) = \alpha f(x) + \beta g(x)$$

Theorem 2.11 (Basic Formula) Let a be a constant and m be an integer. Then

$$\frac{1}{D-a}f(x) = e^{ax}\int e^{-ax}f(x)\,dx$$

$$\frac{1}{(D-a)^m}f(x) = e^{ax}\iint\cdots\int e^{-ax}f(x)\,dx\cdots dxdx \text{ (m repeated integrals)}$$

Proof Let $(D-a)^{-1}f(x) = y$. Then $(D-a)y = f(x)$. Thus

$$\frac{dy}{dx} - ay = f(x).$$

This is a linear differential equation. Thus the integrating factor $\mu = e^{\int(-a)dx}$. Multiply the μ both sides

$$d(e^{\int(-a)dx}y) = e^{\int(-a)dx}f(x).$$

Thus

$$y = e^{-\int(-a)dx}\int e^{\int(-a)dx}f(x)\,dx = e^{ax}\int e^{-ax}f(x)\,dx$$

Repeating this

$$\begin{aligned}
\frac{1}{(D-a)^2} &= \frac{1}{D-a}\Big(\frac{1}{D-a}f(x)\Big) = e^{ax}\int e^{-ax}\Big(\frac{1}{D-a}f(x)\Big)\,dx \\
&= e^{ax}\int e^{-ax}\Big(e^{ax}\int e^{-ax}f(x)\Big)\,dx \\
&= e^{ax}\iint e^{-ax}f(x)\,dx
\end{aligned}$$

Theorem 2.12 (Differential Operator Properties)

$$
\begin{aligned}
(1)\ f(x) &= e^{ax} \Longrightarrow \frac{1}{L(D)}e^{ax} = \frac{1}{L(a)}e^{ax} \\
(2)\ f(x) &= x^m \Longrightarrow \frac{1}{1-aD}x^m = \big(1+(aD)+(aD)^2+\cdots(aD)^m\big)x^m \\
(3)\ f(x) &= \cos ax \Longrightarrow \frac{1}{L(D^2)}\cos ax = \frac{1}{L(-a^2)}\cos ax \\
(4)\ f(x) &= \sin ax \Longrightarrow \frac{1}{L(D^2)}\sin ax = \frac{1}{L(-a^2)}\sin ax \\
(5)\ f(x) &= e^{ax}u(x) \Longrightarrow \frac{1}{L(D)}e^{ax}u(x) = e^{ax}\frac{1}{L(D+a)}u(x) \\
(6)\ f(x) &= x^m u(x) \Longrightarrow \frac{1}{L(D)}x^m u(x) = \sum_{k=0}^{m}\binom{m}{k}x^{m-k}\Big(\frac{1}{L(D)}\Big)^{(k)}u(x)
\end{aligned}
$$

Example 2.20 Solve the differential equation $L(y) = y'' - y' - 2y = 2e^{3x}$.

SOLUTION The characteristic equation of $L(y) = 0$ is $m^2 - m - 2 = 0$. Thus roots are $m = -1, 2$. Then the complementary solution y_c is given by

$$y_c = c_1e^{-x} + c_2e^{2x}$$

Now we find the particular solution. Since $(D^2 - D - 2)y = 2e^{3x}$, by (1) above

$$y = \frac{2e^{3x}}{D^2 - D - 1} = \frac{2e^{3x}}{3^2 - 3 - 1} = \frac{2e^{3x}}{4} = \frac{e^{3x}}{2}.$$

Thus $y_p = \frac{1}{2}e^{3x}$ and the general solution is

$$y = y_c + y_p = c_1e^{-x} + c_2e^{2x} + \frac{1}{2}e^{3x} \ \blacksquare$$

Example 2.21 Solve the differential equation $L(y) = y'' + 4y = \sin x$.

SOLUTION The characteristic equation of $L(y) = 0$ is $m^2 + 4 = 0$. Thus $m = \pm 2i$. Then the complementary solution is

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

Now we find the particular solution. Since $f(x) = \sin x$, we have $(D^2 + 4)y = \sin x$ and

$$y = \frac{\sin x}{D^2 + 4} = \frac{\sin x}{-1^2 + 4} = \frac{1}{3}\sin x.$$

Example 2.22 Find the particular solution of $L(y) = y'' - 3y' + 2y = xe^x$

SOLUTION Let y_p be the particular solution. Then since $f(x) = xe^x$, we have

$$\begin{aligned} y_p &= \frac{1}{D^2 - 3D + 2}(xe^x) = e^x \frac{1}{(D+1)^2 - 3(D+1) + 2} x = e^x \frac{1}{D^2 - D} x \\ &= e^x \frac{1}{D} \cdot \frac{1}{D-1} x = -e^x \frac{1}{D} \frac{1}{1-D} x \\ &= -e^x \frac{1}{D}(1 + D + D^2 + \cdots)x = -e^x \frac{1}{D}(x+1) = -e^x \big(\frac{1}{2}x^2 + x\big) \end{aligned}$$

Chapter 3

Systems of differential equations

3.1 Systems of linear differential equation

system of linear differential equation

Consider

$$\begin{cases} x_1' &= a_{11}x_1 + a_{12}x_2 + & \cdots + a_{1n}x_n + f_1(t) \\ x_2' &= a_{21}x_1 + a_{22}x_2 + & \cdots + a_{2n}x_n + f_2(t) \\ \vdots & \quad\quad \vdots & \quad\quad \vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + & \cdots + a_{nn}x_n + f_n(t) \end{cases}$$

Using the coefficient matrix A and the column vector X anf F, we can express

$$\mathbf{X}' = A\mathbf{X} + \mathbf{F}$$

where

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \ \mathbf{F} = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

The differentiable functions $x_1, x_2, \ldots, x_n$ satisfying the above equation is called the **solution**. When $\mathbf{F} = \mathbf{0}$, we say that the differential equation is **homogeneous equation**.

How to solve $\mathbf{X}' = A\mathbf{X}$

We will explain how to solve $\mathbf{X}' = A\mathbf{X}$. Let $\mathbf{X} = \mathbf{C}e^{\lambda x}$. Then

$$\lambda \mathbf{C}e^{\lambda x} = A\mathbf{C}e^{\lambda x}$$

Since $e^{\lambda x} \neq 0$, we have

$$A\mathbf{C} - \mathbf{C}\lambda = \mathbf{0}$$

or

$$(A - \lambda I)\mathbf{C} = \mathbf{0}$$

Here I denotes the unit matrix of the degree n. Now if $\mathbf{C} = \mathbf{0}$, then the equation is obviously satisfied. So, we need to find the $\mathbf{C} \neq \mathbf{0}$ satisfying the above equation. Note that if $\det(A - \lambda I) \neq 0$, then $\mathbf{C} = 0$. Thus to find the $\mathbf{C} \neq \mathbf{0}$, we have to solve $\det(A - \lambda I) = 0$. The λ is called the **eigenvalue** of the matrix A and the nonzero $\mathbf{C}$ satisfying $(A - \lambda I)\mathbf{C} = \mathbf{0}$ is called the **eigenvector** for A.

Example 3.1 Solve the following differential equation

$$\mathbf{X}' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \mathbf{X}$$

SOLUTION Let $\mathbf{X} = \mathbf{C}e^{\lambda t}$. Then we have

$$(A - \lambda I)\mathbf{C} = \mathbf{0} \ (*)$$

Now

$$\det(A - \lambda I) = \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -1 \\ 1 & 0 & 1-\lambda \end{pmatrix} = \lambda(1-\lambda)(\lambda - 2)$$

Thus the eigenvalues are $\lambda = 0, 1, 2$. Now we find the eigenvector for $\lambda = 0$. Substitute $\lambda = 0$ into the equation (*). Then

$$A\mathbf{C} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \mathbf{0}$$

To solve this system, we use **Gaussian elimination**.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \overset{-R_1+R_3}{\longrightarrow} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we can let $c_3 = 1$. Then $c_1 = -1$, $c_2 = 1$ Thus, the eigenvector $\mathbf{C}$ is $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{X}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is a solution.

For the eigenvector for $\lambda = 1$.

$$\begin{aligned} A - I &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \overset{R_1 \leftrightarrow R_3}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \\ &\overset{R_2+R_3}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Thus, we can take $c_2 = 1$. Then $c_1 = 0$, $C_3 = 0$ and the eigenvector is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. $\mathbf{X}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^t$ is a solution. Similarly, we find the eigenvector corresponds to $\lambda = 2$

$$A - 2I = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \overset{\substack{-R_1 \\ R_1+R_3 \\ -R_2}}{\longrightarrow} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Then we can take $c_3 = 1$ and $c_1 = 1, c_2 = -1$. Thus the eigenvector is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. $\mathbf{X}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{2t}$ is a solution. Since $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are linearly independent, we have $\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + c_3\mathbf{X}_3$ and the general solution is

$$\mathbf{X} = c_1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^t + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{2t} \ \blacksquare$$

Theorem 3.1 Suppose that $\mathbf{X}' = A\mathbf{X}$ has the n different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n$. Then the general solution is given by

$$\mathbf{X}(t) = \sum_{i=1}^{n} c_i \mathbf{X}_i(t)$$

where $\mathbf{X}_i = \mathbf{C}_i e^{\lambda_i t}$ $(i = 1, 2, \ldots, n)$ are the solutions of $\mathbf{X}' = A\mathbf{X}$ and linearly independent.

Proof

$$\begin{aligned} W(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots e^{\lambda_n t}) &= \begin{vmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} & \ldots & e^{\lambda_n t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} & \ldots & \lambda_n e^{\lambda_n t} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} e^{\lambda_1 t} & \lambda_2^{n-1} e^{\lambda_2 t} & \ldots & \lambda_n^{n-1} e^{\lambda_n t} \end{vmatrix} \\ &= e^{\lambda_1 t} e^{\lambda_2 t} \cdots e^{\lambda_n t} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1} \end{vmatrix} \\ &= \Pi_{1 \le i < j \le n-1} (\lambda_j - \lambda_i) \end{aligned}$$

Since $\lambda_1, \lambda_2, \ldots \lambda_n$ are different,

$$W(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}) \neq 0 \ \blacksquare$$

The n linearly independent solutions of $\mathbf{X}' = A\mathbf{X}$ is called the **fundamental solution**.

Example 3.2 Solve the following differential equation

$$\begin{cases} x_1' &= 8x_1 + 6x_2 \\ x_2' &= -3x_1 - 2x_2 \end{cases}.$$

SOLUTION

$$\begin{aligned} x_1' &= 8x_1 + 6x_1 \\ x_2' &= -3x_1 - 2x_2 \end{aligned}$$

or

$$\mathbf{X}' = \begin{pmatrix} 8 & 6 \\ -3 & -2 \end{pmatrix} \mathbf{X}$$

We find the eigenvalues and eigenvectors.

$$\begin{vmatrix} 8-\lambda & 6 \\ -3 & -2-\lambda \end{vmatrix} = -16 - 6\lambda + \lambda^2 + 18 = \lambda^2 - 6\lambda + 2 = 0$$

Thus we have $\lambda = 3 \pm \sqrt{7}$. For $\lambda_1 = 3 + \sqrt{7}, \mathbf{C}_1 = \begin{pmatrix} -6 \\ 5 - \sqrt{7} \end{pmatrix}$. For $\lambda_2 = 3 - \sqrt{7}, \mathbf{C}_2 = \begin{pmatrix} -6 \\ 5 + \sqrt{7} \end{pmatrix}$. Thus the general solution is

$$\mathbf{X}(t) = c_1 \begin{pmatrix} -6 \\ 5 - \sqrt{7} \end{pmatrix} e^{(3+\sqrt{7})t} + c_2 \begin{pmatrix} -6 \\ 5 + \sqrt{7} \end{pmatrix} e^{(3-\sqrt{7})t} \blacksquare$$

3.1.1 Exercise

1. Solve the following systems of differential equations.

(a) $\begin{cases} x_1' = & x_1 + 4x_2 \\ x_2' = & x_1 + x_2 \end{cases}$ (b) $\begin{cases} x_1' = & 6x_1 - 3x_2 \\ x_2' = & 2x_1 + x_2 \end{cases}$

(c) $\begin{cases} x_1' = & x_1 + x_2 - x_3 \\ x_2' = & 2x_2 \\ x_3' = & x_2 - x_3 \end{cases}$ (d) $\mathbf{X}' = \begin{pmatrix} 1 & -3 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & -2 \end{pmatrix} \mathbf{X}$

(e) $\begin{cases} x_1' + x_2' + 2x_2 = & 0 \\ x_1' - 3x_1 - 2x_2 = & 0 \end{cases}$ (f) $\begin{cases} x_1' + 6x_1 + x_2' + 3x_2 = & 0 \\ x_1' - x_2' + x_2 = & 0 \end{cases}$

3.2 Multiple roots and Complex roots

In the previous section, we consider the case where the eigen equation has the distinct real roots. In this section, we consider the case of multiple roots and complex roots.

Complex roots

Let $\mathbf{X}' = A\mathbf{X}$, where the matrix A is real matrix. Suppose that $\lambda = a + bi$ is the eigenvalue and $\mathbf{C} = \mathbf{C}_1 + i\mathbf{C}_2$ ($\mathbf{C}_1, \mathbf{C}_2$real vector) is the eigenvector for λ. Then by the **eigenvalue equation**

$$(A - \lambda I)\mathbf{C} = \mathbf{0}$$

The conjugate of the eigenvalue equation satisfies

$$\overline{(A - \lambda I)\mathbf{C}} = (\bar{A} - \bar{\lambda}\bar{I})\bar{\mathbf{C}} = \mathbf{0}$$

Thus, $\bar{\lambda} = a - bi$ is also an eigenvalue and corresponding eigenvector is $\bar{\mathbf{C}}$. Thus, $\mathbf{X}_1 = \mathbf{C}e^{\lambda t}$ and $\mathbf{X}_2 = \bar{\mathbf{C}}e^{\bar{\lambda}t}$ are solutions of $\mathbf{X}' = A\mathbf{X}$. Then the linear combination of $\mathbf{X}_1$ and $\mathbf{X}_2$ is also a solution. Therefore,

$$\Re\mathbf{C}e^{\lambda t} = \Re\mathbf{X}_1 = \frac{\mathbf{X}_1 + \mathbf{X}_2}{2} = \mathbf{C}_1 e^{at}\cos bt - \mathbf{C}_2 e^{at}\sin bt$$

and

$$\Im\mathbf{C}e^{\lambda t} = \Im\mathbf{X}_1 = \frac{\mathbf{X}_1 - \mathbf{X}_2}{2i} = \mathbf{C}_1 e^{at}\sin bt - \mathbf{C}_2 e^{at}\cos bt$$

are linearly independent solutions.

Example 3.3 Solve the following differential equation

$$\mathbf{X}' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -2 \\ 2 & 2 & 0 \end{pmatrix}\mathbf{X}.$$

SOLUTION

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & -2 \\ 2 & 2 & -\lambda \end{pmatrix} = (1-\lambda)(\lambda^2 + 4) = 0$$

Thus we have $\lambda = 1, \pm 2i$.

For $\lambda = 1$, we find the eigenvector.

$$A - I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & -2 \\ 2 & 2 & -1 \end{pmatrix} \overset{-2R_2+R_3}{\underset{}{\overset{R_2 \leftrightarrow R_1}{\longrightarrow}}} \begin{pmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 4 & 3 \end{pmatrix}$$

$$\overset{\frac{1}{4}R_2}{\overset{\frac{1}{4}R_2+R_1}{\overset{R_2 \leftrightarrow R_3}{\longrightarrow}}} \begin{pmatrix} 1 & 0 & -5/4 \\ 0 & 1 & 3/4 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that we c_3 can be chosen arbitrary. Thus we let $c_3 = 4$. Then the eigenvector is $\mathbf{C} = \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix}$.

For $2i$, we find the corresponding eigenvector

$$\begin{aligned}
A-2iI &= \begin{pmatrix} 1+2i & 0 & 0 \\ 1 & 2i & -2 \\ 2 & 2 & -2i \end{pmatrix} \\
&\xrightarrow[R_1\leftrightarrow R_2]{-2R_2+R_3} \begin{pmatrix} 1 & 2i & -2 \\ 1+2i & 0 & 0 \\ 0 & 2-4i & 4+2i \end{pmatrix} \\
&\xrightarrow[\frac{-2i}{4-2i}R_2+R_1]{\substack{-(1+2i)R_1+R_2 \\ \frac{1}{4-2i}R_2 \\ \frac{-(2-4i)}{4-2i}R_2+R_3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Now c_3 is free to choose. Thus the eigenvector is $\mathbf{C} = \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$. Now we find the real part and imaginary part of $\mathbf{C}e^{\lambda t}$Then

$$\begin{aligned}
\mathbf{C}e^{\lambda t} &= \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} e^{-2it} = \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} (\cos 2t - i \sin 2t) \\
&= \begin{pmatrix} 0 \\ \sin 2t - i\cos 2t \\ \cos 2t - i \sin 2t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ \sin 2t \\ \cos 2t \end{pmatrix}}_{\text{real part}} + i \underbrace{\begin{pmatrix} 0 \\ -\cos 2t \\ -\sin 2t \end{pmatrix}}_{\text{imaginary part}}
\end{aligned}$$

Thus the general solution is

$$\mathbf{X}(t) = c_1 \begin{pmatrix} 5 \\ -3 \\ -e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ \sin 2t \\ \cos 2t \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -\cos 2t \\ -\sin 2t \end{pmatrix}. \blacksquare$$

Multiple roots

Let $\mathbf{X}' = A\mathbf{X}$, where the matrix A is real matrix. Suppose that λ is multiple eigenvalues and A is not diagonalizable. Then cosider $\mathbf{X} = \mathbf{C}e^{At}$. Since

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots,$$

we have the exponential matrix e^{At} such that

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

Thus

$$\frac{d}{dt}e^{At} = A + tA^2 + \frac{t^2}{2!}A^3 + \cdots = Ae^{At}$$

and $\mathbf{X} = \mathbf{C}e^{At}$ is a solution of $\mathbf{X}' = A\mathbf{X}$.

How to find $\mathbf{C}e^{At}$

$$\begin{aligned}\mathbf{C}e^{At} &= \mathbf{C}e^{(A-\lambda I)t}e^{\lambda t} = e^{\lambda t}(\mathbf{C} + t(A-\lambda I)\mathbf{C} + \frac{t^2}{2!}(A-\lambda I)^2\mathbf{C} \\ &+ \frac{t^3}{3!}(A-\lambda I)^3\mathbf{C} + \cdots + \frac{t^k}{k!}(A-\lambda I)^k\mathbf{C} + \cdots)\end{aligned}$$

Now note that if k is the least number satisfing $(A-\lambda I)^k\mathbf{C} = 0$, then

$$\mathbf{C}e^{At} = e^{\lambda t}(\mathbf{C} + t(A-\lambda I)\mathbf{C} + \frac{t^2}{2!}(A-\lambda I)^2\mathbf{C} + \cdots + \frac{t^{k-1}}{(k-1)!}(A-\lambda I)^{(k-1)}\mathbf{C})$$

Example 3.4 Solve the following differential equation

$$\begin{cases} x_1' = & 2x_1 - x_2 + 2x_3 \\ x_2' = & 2x_2 + 2x_3 \\ x_3' = & 2x_3 \end{cases}$$

SOLUTION Since

$$\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & -1 & 2 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)^3$$

then eigenvalues are $\lambda = 2$. Now we find the eigenvector $\mathbf{C}$ for $\lambda = 2$.

$$A - 2I = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \overset{\substack{-1\times R_1 \to R_1 \\ \frac{1}{2}\times R_2 \to R_2}}{\longrightarrow} \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \overset{2R_2 + R_3 \to R_3}{\longrightarrow} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $\mathbf{C} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and the eigenvector is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$$

Since the degree of the matrix A is 3, we have to find three linearly independent solutions. Thus we need to find $\mathbf{C}$ such that

$$(A - 2I)^2\mathbf{C} = \mathbf{0}, \ (A - 2I)\mathbf{C} \neq \mathbf{0}$$

Since

$$(A - 2I)^2 = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

c_1, c_2 can be chosen arbitrary. Thus we let $\mathbf{C} = \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}$. Then $(A - 2I)^2\mathbf{C} = \mathbf{0}$. Now we choose α, β such that $(A - 2I)\mathbf{C} \neq \mathbf{0}$.

$$(A - 2I)\mathbf{C} = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} = \begin{pmatrix} -\beta \\ 0 \\ 0 \end{pmatrix}.$$

Then we choose $\beta = 1$. $\mathbf{C} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and the second solution is

$$\begin{aligned} e^{At}\mathbf{C} &= e^{2t}e^{(A-2I)t}\mathbf{C} = e^{2t}[\mathbf{C} + t(A - 2I)\mathbf{C}] = e^{2t}\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\} \\ &= e^{2t}\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -t \\ 0 \\ 0 \end{pmatrix}\} = e^{2t}\begin{pmatrix} 1 - t \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

For the third solution, we need to find $\mathbf{C}$ satisfying

$$(A - 2I)^3\mathbf{C} = \mathbf{0}, \ (A - 2I)^2\mathbf{C} \neq \mathbf{0}$$

Note that $(A - 2I)^3 = O$. Then $\mathbf{C} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ satisfies $(A - 4I)^3\mathbf{C} = \mathbf{0}$. Now

$$(A - 2I)^2\mathbf{C} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -2\gamma \\ 0 \\ 0 \end{pmatrix}.$$

Thus we choose $\gamma = 1$. Then $\mathbf{C} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ Now calculate the third solution

$$\begin{aligned}
e^{At}\mathbf{C} &= e^{2t}e^{(A-2I)t}\mathbf{C} \\
&= e^{2t}[\mathbf{C} + t(A-2I)\mathbf{C} + \frac{t^2}{2!}(A-2I)^2\mathbf{C}] \\
&= e^{2t}[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}] \\
&= e^{2t}\begin{pmatrix} 2t - t^2 \\ 2t \\ 1 \end{pmatrix}
\end{aligned}$$

Note that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1-t \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2t-t^2 \\ 2t \\ 1 \end{pmatrix}$ are linearly independent. Thus

$$\mathbf{X} = e^{2t}[c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1-t \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 2t-t^2 \\ 2t \\ 1 \end{pmatrix}] \blacksquare$$

3.2.1 Exercise

1. Solve the following differential equation.

(a) $\begin{cases} x_1' = & 6x_1 + 8x_2 \\ x_2' = & -x_1 + 2x_2 \end{cases}$ (b) $\begin{cases} x_1' = & 2x_1 - x_2 \\ x_2' = & 4x_1 + 6x_2 \end{cases}$

(c) $\begin{cases} x_1' = & 5x_1 + 2x_2 + 2x_3 \\ x_2' = & 2x_1 + 2x_2 - 4x_3 \\ x_3' = & 2x_1 - 4x_2 + 2x_3 \end{cases}$ (d) $\mathbf{X}' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix}\mathbf{X}$

(e) $\begin{cases} 4x_1' + x_1 + 2x_2' + 7x_2 = & 0 \\ x_1' - x_1 + x_2' + x_2 = & 0 \end{cases}$ (f) $\begin{cases} x_1' + x_1 + 2x_2' + 3x_2 = & 0 \\ x_1' - 2x_1 + 5x_2' = & 0 \end{cases}$

3.3 Nonhomogeneous differential equations

Solutions of nonhomogeneous differential equations

Let A be the square matrix of order n. Suppose that $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are linearly independent solutions of $\mathbf{X}' = A\mathbf{X}$. Then

$$\Phi = \begin{pmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \cdots & \mathbf{X}_n \end{pmatrix}$$

is called the **fundamental matrix**. Now
1. $\det \Phi = W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) \neq 0$ (W = Wronskian)
2. $\Phi' = A\Phi$
3. $c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n = \Phi \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \Phi\mathbf{C}$

Thus the complementary solution of $\mathbf{X}' = A\mathbf{X}$ is given by $\Phi\mathbf{C}$. Now using the variation of parameter, we let $\Phi(t)\mathbf{U}(t)$ be the solution of $\mathbf{X}' = A\mathbf{X} + \mathbf{F}$. Then since the derivative of the vector valued function is given by the derivatives of components of the vector valued function, we have

$$(\Phi\mathbf{U})' = \Phi'\mathbf{U} + \Phi\mathbf{U}'$$

Now substitute $\mathbf{X} = \Phi\mathbf{U}$ into $\mathbf{X}' = A\mathbf{X} + \mathbf{F}$. Then

$$\Phi'\mathbf{U} + \Phi\mathbf{U}' = A\Phi\mathbf{U} + \mathbf{F}$$

Since $\Phi' = A\Phi$, we have

$$\Phi\mathbf{U}' = \mathbf{F}$$

Then by the Cramer's rule

$$\mathbf{U}' = \frac{[\Phi : \mathbf{F}]}{|\Phi|}$$

Integrate to get the general solution

$$\mathbf{X} = \Phi\mathbf{C} + \Phi\mathbf{U}$$

Example 3.5 Solve the differential equation $\mathbf{X}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\mathbf{X} + \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$

SOLUTION $\det(A - \lambda I) = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4) = 0$. Thus the eigenvalues are $\lambda = -1, 4$. The eigenvector corresponds to $\lambda = -1$ is obtained by $(A - (-1(I) = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$. Then the eigenvector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Thus $\mathbf{X}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$ is a solution. The eigenvector corresponds to $\lambda = 4$ is obtained by $(A - 4I) = \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}$. Thus the eigenvector is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Therefore, $\mathbf{X}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t}$ is a solution. Then the fundamental matrix is

$$\Phi(t) = \begin{pmatrix} -e^{-t} & 2e^{4t} \\ e^{-t} & 3e^{4t} \end{pmatrix}$$

. Now to find the general solution, we need to solve

$$\Phi\mathbf{U}' = \mathbf{F}$$

$$\begin{pmatrix} -e^{-t} & 2e^{4t} \\ e^{-t} & 3e^{4t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$$

By Cramer's rule,

$$u_1' = \frac{\begin{vmatrix} e^{3t} & 2e^{4t} \\ 2e^{3t} & 3e^{4t} \end{vmatrix}}{\begin{vmatrix} -e^{-t} & 2e^{4t} \\ e^{-t} & 3e^{4t} \end{vmatrix}} = \frac{-e^{7t}}{-5e^{3t}} = \frac{1}{5}e^{4t}$$

$$u_2' = \frac{\begin{vmatrix} -e^{-t} & e^{3t} \\ e^{-t} & 2e^{3t} \end{vmatrix}}{-5e^{3t}} = \frac{-3e^{2t}}{-5e^{3t}} = \frac{3}{5}e^{-t}$$

Integrating,

$$u_1 = \frac{1}{20}e^{4t}, \; u_2 = -\frac{3}{5}e^{-t}$$

Thus the general solution is

$$\begin{aligned} \mathbf{X} &= \Phi\mathbf{C} + \Phi\mathbf{U} = \begin{pmatrix} -e^{-t} & 2e^{4t} \\ e^{-t} & 3e^{4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -e^{-t} & 2e^{4t} \\ e^{-t} & 3e^{4t} \end{pmatrix} \begin{pmatrix} \frac{1}{20}e^{4t} \\ -\frac{3}{5}e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} -c_1e^{-t} + 2c_2e^{4t} - \frac{5}{4}e^{3t} \\ c_1e^{-t} + 3c_2e^{4t} - \frac{7}{4}e^{3t} \end{pmatrix} \end{aligned}$$

■

Elimination Method

Let $D = d/dt$. Then $x_1' = Dx_1$ and the given differential equation

$$\mathbf{X}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$$

can be written as

$$\begin{cases} (D-1)x_1 - 2x_2 &= e^{3t} \\ -3x_1 + (D-2)x_2 &= 2e^{3t} \end{cases}$$

Then solve for x_1 by using Cramer's rule.

Example 3.6 Solve the following differential equation

$$\begin{cases} (D-1)x_1 - 2x_2 &= e^{3t} \\ -3x_1 + (D-2)x_2 &= 2e^{3t} \end{cases} \tag{3.1}$$

SOLUTION Using Cramer's rule, we have

$$\begin{vmatrix} D-1 & -2 \\ -3 & D-2 \end{vmatrix} x_1 = \begin{vmatrix} e^{3t} & -2 \\ 2e^{3t} & D-2 \end{vmatrix}.$$

Then

$$(D^2 - 3D - 4)x_1 = (D-2)e^{3t} + 2(2e^{3t}) = 5e^{3t}.$$

The characteristic equation of the equation is $m^2 - 3m - 4 = 0$. Thus $m = -1, 4$ and the complementary solution x_{1c} is

$$x_{1c} = c_1e^{-t} + c_2e^{4t}.$$

To find x_{1p}, we use the method of undetermined coefficients

$$H(D)5e^{3t} = (D-3)5e^{3t} = 0$$

implies that

$$(D-3)(D^2-3D-4)x_1 = (D-3)5e^{3t} = 0.$$

Thus, $x_{1p} = Ae^{3t}$.

$$9Ae^{3t} - 9Ae^{3t} - 4Ae^{3t} = 5e^{3t}.$$

Solve this to get $A = -\frac{5}{4}$. Then

$$x_1 = x_{1c} + x_{1p} = c_1e^{-t} + c_2e^{4t} - \frac{5}{4}e^{3t}.$$

Now put this back to (3.1).

$$-c_1e^{-t} + 4c_2e^{4t} - \frac{15}{4}e^{3t} - c_1e^{-t} - c_2e^{4t} + \frac{5}{4}e^{3t} - 2x_2 = e^{3t}.$$

Thus,

$$x_2 = -c_1e^{-t} + \frac{3}{2}c_2e^{4t} - \frac{7}{4}e^{2t} \blacksquare$$

Example 3.7 Solve the following differential equation

$$\begin{cases} x_1'' - 2x_1 - 3x_2 = & 0 \\ x_1 + x_2'' + 2x_2 = & 0 \end{cases}$$

SOLUTION Let $u = x_1'$ and $v = x_2'$. Then $u' = x_1''$ and $v' = x_2''$. Thus we can express

$$\begin{pmatrix} x_1 \\ x_2 \\ u \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 3 & 0 & 0 \\ -1 & -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ u \\ v \end{pmatrix}$$

$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 3 & 0 & 0 \\ -1 & -2 & 0 & 0 \end{pmatrix}$ Then the eigeneqution is $\lambda^4 - 1 = 0$ and the eigenvalues are $\lambda = \pm 1, \pm i$. To find the eigenvector, we use Mathematica. Then the eigenvectors correspond to $\lambda = 1, -1, i$ are

$$(3, -1, -3, 1)^t, (-3, 1, -3, 1)^t, (-i, i, -1, 1)^t$$

Thus $\mathbf{C}e^{\lambda t}$ is

$$\begin{pmatrix} 3e^t \\ -e^t \\ -3e^t \\ e^t \end{pmatrix}, \begin{pmatrix} -3e^{-t} \\ e^{-t} \\ -3e^{-t} \\ e^t \end{pmatrix}, \begin{pmatrix} -i(\cos t + i\sin t) \\ i(\cos t + i\sin t) \\ -(\cos t + i\sin t) \\ \cos t + i\sin t \end{pmatrix}$$

Thus the fundamental matric $\Phi(t)$ is

$$\Phi(t) = \begin{pmatrix} 3e^t & -3e^{-t} & \sin t & -\cos t \\ -e^t & e^{-t} & -\sin t & \cos t \\ -3e^t & -3e^{-t} & -\cos t & \sin t \\ e^t & e^t & \cos t & \sin t \end{pmatrix}$$

Therefore the general solution is

$$\mathbf{X} = \begin{pmatrix} 3e^t & -3e^{-t} & \sin t & -\cos t \\ -e^t & e^{-t} & -\sin t & \cos t \\ -3e^t & -3e^{-t} & -\cos t & \sin t \\ e^t & e^t & \cos t & \sin t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

$$\begin{aligned} x_1 &= c_1 3e^t - c_2 3e^t = c_3 \sin t - c_4 \cos t \\ x_2 &= -c_1 e^t + c_2 e^{-t} - c_3 \sin t + c_4 \cos t \ \blacksquare \end{aligned}$$

3.3.1 Exercise

1. Solve the following differential equations.

(a) $\begin{cases} x_1' = 2x_1 + x_2 - e^t \\ x_2' = 3x_1 + 4x_2 - 7e^t \end{cases}$ (b) $\mathbf{X}' = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \mathbf{X} + \begin{pmatrix} -3\cos t \\ 0 \end{pmatrix}$

(c) $\begin{cases} x_1' = x_1 + x_2 + x_3 + 1 \\ x_2' = -x_2 \\ x_3' = -2x_1 - x_2 - 2x_3 + \frac{3}{e^t} \end{cases}$ (d) $\mathbf{X}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \mathbf{X} + \begin{pmatrix} t^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

2. Solve the differential equation by using the system of differential equation.
$y'' + 4y' + 3y = t$

3. Solve by elimination.

(a) $\begin{cases} x_1' + x_2' - 2x_1 - 4x_2 &= e^t \\ x_1' + x_2' - x_2 &= e^{4t} \end{cases}$ (b) $\begin{cases} x_1'' + x_2' - x_1 + x_2 &= 1 \\ x_1' + x_2'' - x_1 + x_2 &= 0 \end{cases}$

(c) $\begin{cases} 2x_1' + x_2' + x_1 + 5x_2 &= 4t \\ x_1' + x_2' + 2x_1 + 2x_2 &= 2 \end{cases}$ (d) $\begin{cases} x_1'' - x_2' &= t + 1 \\ x_1' + x_2' - 3x_1 + x_2 &= 2t - 1 \end{cases}$

Chapter 4

Power series solution of differential equation

4.1 Analytic functions

Let $\{a_n(x)\}_{n=1}^{\infty}$ be a sequence of functions defined on the interval I. Then consider the partial sums

$$S_1(x) = a_1(x), S_2(x) = a_1(x) + a_2(x), S_3(x) = a_1(x) + a_2(x) + a_3(x), \ldots$$

If $\lim_{n\to\infty} S_n(x) = S(x)$ for all x in I, then we say $\{S_n(x)\}_{n=1}^{\infty}$ converges and $S(x)$ is called the sum of the series . For all x in I, there exists N such that

$$n \geq N \text{implies } |S_n(x) - S(x)| < \epsilon$$

Then $\sum a_n(x)$ is called **uniformly convergent**.

Let $S(x) - S_n(x) = R(x)$. Then the uniform convergence of the series is expressed as follows: For each x in the interval I, there exists a number N such that $n \geq N implies that |R(x)| < \epsilon$.

Theorem 4.1 (Weierstrass uniformly convergent criterion) For any x in some interval I, there exists a sequence of real functions $\{M_n\}$ satisfying the followings
(1) $|a_n(x)| \leq M_n, \ n = 1, 2, \ldots$
(2) $\sum M_n < \infty$
Then $\sum a_n(x)$ converges uniformly and absolutely on I.

Proof

$$\begin{aligned} |R_n(x)| &= |a_{n+1}(x) + a_{n+2}(x) + \cdots| \leq |a_{n+1}(x)| + |a_{n+2}(x)| + \cdots \\ &\leq M_{n+1} + M_{n+2} + \cdots \end{aligned}$$

Since $\sum M_n < \infty$, for all $\epsilon > 0$, there exists N such that $n \geq N$ implies

$$M_{n+1} + M_{n+2} + \cdots < \epsilon.$$

Here N is indepedent of x, for $n > N$, $|R_n(x)| < \epsilon$ ■

Theorem 4.2 (Continuity of power series) If $a_n(x)$ is continuous on the interval I and $\sum a_n(x)$ converges uniformly on I, then $S(x) = \sum a_n(x)$ is continuous on I.

Theorem 4.3 (Term-by-term integration of series of functions) If $a_n(x)$ is continuous on the interval I and $\sum a_n(x)$ is uniformly convergent on I, then

$$\int (\sum a_n(x))dx = \sum \int a_n(x)dx$$

Theorem 4.4 (Term-by-term differentiation of series of functions) If $a_n(x)$ is continuous on the interval I and $\sum a_n(x)$ is uniformly convergent on I, then

$$(\sum a_n(x))' = \sum a_n'(x)$$

If $a_n(x) = a_n x^n$, then the series $\sum a_n(x) = \sum a_n x^n$ is called a **power series**. For example,

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

Radius of convergence

Given $\sum a_n x^n$, by the ratio test, if

$$\lim_{n\to\infty} \mid \frac{c_{n+1}}{c_n} \mid = \lim_{n\to\infty} \mid \frac{a_{n+1}x^{n+1}}{a_n x^n} \mid = |x| \lim_{n\to\infty} \mid \frac{a_{n+1}}{a_n} \mid < 1,$$

then $\sum a_n x^n$ converges. Thus, if $|x| < \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}|$, then $\sum |a_n x^n| < \infty$
If $|x| > \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}|$, then $\sum a_n x^n$ is divergent. Then the limit

$$\rho = \lim_{n\to\infty} \mid \frac{a_n}{a_{n+1}} \mid$$

is called the **radius of convergence**

Example 4.1 Find the radius of convergence of the following power series
$(a)\ \sum \frac{x^n}{n^2}$ $\quad (b)\ \sum \frac{x^n}{n!}$

SOLUTION

(a) $\rho = \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}| = \lim_{n\to\infty} |\frac{(n+1)^2}{n^2}| = 1$

(b) $\rho = \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}| = \lim_{n\to\infty} |\frac{(n+1)!}{n!}| = \infty.$

Theorem 4.5 (Properties of power series) Let *rho* be the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$. Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then
(1) For all r $(0 < r < \rho)$, $\sum a_n x^n$ converges uniformly on $(-r, r)$.
(2) $f(x)$ is the class C^∞ on $(-\rho, \rho)$.
(3) $f(x)$ is continuous on $(-\rho, \rho)$.
(4) $f(x)$ is term-by-term differentiable on $(-\rho, \rho)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

(5) $f(x)$ is term-by-term integrable on $(-\rho, \rho)$ and

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

(6) Taylor series of $f(x)$ on $(-\rho, \rho)$ is the same as the power series $\sum_{n=0}^{\infty} a_n x^n$. Thus,

$$a_n = \frac{f^{(n)}(0)}{n!}$$

If $f(x)$ is expressed as the power series of the positive radius $\sum_{n=0}^{\infty} a_n(x-a)^n$, then $f(x)$ is called **analytic** at a.

4.1.1 Exercise

1. Find the radius of convergence of the following power series.
 $(a)\ \sum \frac{nx^n}{3^n}$ $\ (b)\ \sum \frac{n^n x^n}{n!}$
2. Show the following.
 $(a)\ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x},\ |x| < 1$
 $(b)\ \log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1},\ |x| \le 1$
3. Suppose that $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$. Then show the following.
 (a) $f(x)$ is uniformly convergent on $-\infty < x < \infty$.
 (b) $f'(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$
 (c) $\int_0^{\pi} f(x)dx = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^4}$

4.2 Power series solution (Ordinary point)

Consider the 2nd-order linear differential equation with variable coefficients

$$y'' + P(x)y' + Q(x)y = R(x).$$

If $P(x), Q(x)$, and $R(x)$ area analytic at a, then we call a **ordinary point**.

Theorem 4.6 (Existence and uniqueness of initial value problem) For the 2nd-order linear differential equation

$$y'' + P(x)y' + Q(x)y = R(x),$$

if a is an ordinary point, then for any b_0, b_1, the initial value problem

$$y(a) = b_0, \quad y'(a) = b_1$$

has a unique solution $y(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$.

Example 4.2 Find the power series solution around $x = 0$

$$y'' + xy' - 4y = 0.$$

SOLUTION Since $P(x) = x, Q(x) = -4, R(x) = 0$, $x = 0$ is an ordinary point. Now let

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

be a solution. Then,

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

Substitute these back into the given differential equation.

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} nc_n x^n - 4\sum_{n=0}^{\infty} c_n x^n = 0$$

Now let the power of x be the least number $n-2$.,

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=3}^{\infty}(n-2)c_{n-2}x^{n-2} - 4\sum_{n=2}^{\infty} c_{n-2}x^{n-2} = 0.$$

Then let the index be the largest number $n = 3$. Then

$$2c_2 - 4c_0 + \sum_{n=3}^{\infty}[n(n-1)c_n + (n-6)c_{n-2}]x^{n-2} = 0.$$

Since the right-hand side is 0, by the term-by-term differentiation, the coefficients of x^n are all 0. Thus

$$2c_2 - 4c_0 = 0,\ n(n-1)c_n + (n-6)c_{n-2} = 0,\ n \geq 3$$

or

$$c_2 = 2c_0,\ c_n = -\frac{(n-6)c_{n-2}}{n(n-1)},\ n \geq 3.$$

Here, c_0, c_1 can be decided by the initial conditions $y(0), y'(0)$. Thus, we can think of these as constants. Now we find $c_2, c_3, \ldots$. Then

$$c_2 = 2c_0,\ c_3 = \frac{-c_1}{2}, \cdots$$

and

$$c_{2n} = \frac{c_{2n}}{c_{2n-2}}\frac{c_{2n-2}}{c_{2n-4}}\frac{c_{2n-4}}{c_{2n-6}}c_{2n-6}, n \leq 2\ c_{2n} = 0, n \geq 3$$

or

$$c_{2n+1} = \frac{c_{2n+1}}{c_{2n-1}}\frac{c_{2n-1}}{c_{2n-3}}\cdots\frac{c_3}{c_1}c_1 = \frac{3(-1)^n c_1}{2^n n!(2n+1)(2n-1)(2n-3)}x^{2n+1}$$

Thus

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 + 2c_0 x^2 - \frac{c_0}{3}x^4 - c_1\sum_{n=2}^{\infty}\frac{3(-1)^n}{2^n n!(2n+1)(2n-1)(2n-3)}x^{2n+1} \blacksquare \end{aligned}$$

4.2.1 Exercise

1. Find the power series solution of the following differential equation around $x = 0$.
   (a) $y' + y = e^x$ (b) $y'' + xy = 0$
   (c) $xy' - y = x^2 e^x$ (d) $y'' + xy' - 4y = \sin x$
2. Find the power series solution of the following differential equation around the given point.
   (a) $y'' + (x-2)y = 0,\ a = 2$ (b) $y'' + (x-1)y' + 3y = x^2,\ a = 1$

4.3 Frobenius Method (Regular singular point)

For the 2nd-order linear differential equation

$$y'' + P(x)y' + Q(x)y = R(x)$$

if $x = a$ is not an ordinary point, then $x = a$ is called a **singular point**. For $x = a$ singular point and

$$(x-a)P(x),\ (x-a)^2Q(x)$$

are analytic. Then $x = a$ is called a **regular singular point**. Otherwise, **irregular singular point**.

Example 4.3 Find the singular point of the following differential equation

$$(x-1)^2x^2(x-2)y'' + 5x^2y' - (x-2)y = 0.$$

SOLUTION Since

$$P(x) = \frac{5x^2}{(x-1)^2x^2(x-2)},\ Q(x) = \frac{-(x-2)}{(x-1)^2x^2(x-2)}$$

$x = 1, 0, 2$ are singular points. Now

$$(x-1)P(x) = \frac{5x^2}{(x-1)x^2(x-2)}$$

implies that $x = 1$ is irregular singular point and the rest is regular singular point. ■

Theorem 4.7 For the 2nd-order differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

if $x = a$ is the regular singular point, then there exists a solution around $x = a$ of the form

$$y(x) = \sum_{n=0}^{\infty} c_n(x-a)^{n+r},\ c_0 \neq 0$$

The power series $\sum_{n=0}^{\infty} c_n(x-a)^{n+r}$ is called **Frobinius series**. Using the Frobinious series to find c_n is called Frobenius method.

Example 4.4 Find the power series solution of the follwing differential equation around $x = 0$.

$$L(y) =$$

SOLUTION Write $L(y) = x^2y'' + xy' + (x^2-4)y = 0$ as the standard form

$$y'' + \frac{1}{x}y' + \frac{x^2-4}{x^2}y = 0.$$

Then $P(x) = \frac{1}{x},\ Q(x) = \frac{x^2-4}{x^2}$. Thus $x = 0$ is the regular singular point. Now we set a solution of the differential equation as $y = \sum_{n=0}^{\infty} c_nx^{n+r},\ c_0 \neq 0$. Differentiate to get

$$y' = \sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1},\ y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}$$

Substitute these into $L(y) = 0$ to obtain

$$L(y) = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + (x^2-4)\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Now let the power of x be the least number $n+r$. Then

$$\sum_{n=0}^{\infty}(n+r)(n+r)c_n x^{n+r} - \sum_{n=2}^{\infty} c_{n-2}x^{n+r} - 4\sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

Next we let the index be the largest number $n = 2$

$$\underbrace{(r^2-4)c_0 x^r}_{n=0} + \underbrace{((r+1)^2-4)c_1 x^{r+1}}_{n=1} + \sum_{n=2}^{\infty}[((n+r)^2-4)c_n + c_{n-2}]x^{n+r} = 0$$

For $n = 0$, we have the indicial equation $r^2 - 4 = 0$ which implies $r = \pm 2$.

We find a solution for $r = 2$. Note that the coefficients of x^{n+r} are 0. Thus

$$((r+1)^2-4)c_1 = 0$$

and

$$((n+r)^2-4)c_n + c_{n-2} = 0, \; n \geq 2$$

The for $r = 2$, we have $c_1 = 0$. Similarly, for $r = 2$, we have

$$c_n = -\frac{c_{n-2}}{n(n+4)}$$

$$\begin{aligned} c_{2n} &= \frac{c_{2n}}{c_{2n-2}}\frac{c_{2n-2}}{c_{2n-4}}\cdots\frac{c_2}{c_0}c_0 \\ &= \frac{-1}{2n(2n+4)}\frac{-1}{(2n-2)(2n+2)}\cdots\frac{-1}{2\cdot 6}c_0 \\ &= \frac{(-1)^n c_0}{2^n n! 2^{n-1}(n+2)!} = \frac{(-1)^n c_0}{2^{2n-1}n!(n+2)!} \end{aligned}$$

Thus,

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n c_0}{2^{2n-1}n!(n+2)!}x^{2n+2} \blacksquare$$

Example 4.5 Find a solution of the following differential equation around $x = 0$.

$$L(y) = 4x^2 y'' + (3x+1)y = 0.$$

SOLUTION Write in the standard form. Then since $Q(x) = \frac{3x+1}{4x^2}$, $x = 0$ is a regular singular point. We let $y = \sum_{n=0}^{\infty} c_n x^{n+r}$.Then

$$L(\sum c_n x^{n+r}) = \sum_{n=0}^{\infty} 4(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} 3c_n x^{n+r+1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Now let the power of x be the least number x^{n+r}. Then

$$\sum_{n=0}^{\infty}[4(n+r)(n+r-1)+1]c_n x^{n+r} + \sum_{n=1}^{\infty} 3c_{n-1}x^{n+r} = 0.$$

Thus,

$$\underbrace{(4r(r-1)+1)c_0x^r}_{n=0} + \sum_{n=1}^{\infty}\{[4(n+r)(n+r-1)+1]c_n + 3c_{n-1}\}x^{n+r} = 0$$

Now the indicial equation is $4r^2 - 4r + 1 = 0$. Thus $r = \frac{1}{2}$ and the recurrence relation

$$c_n = \frac{-3c_{n-1}}{4(n+r)(n+r-1)+1}$$

can be written as

$$c_n = \frac{-3c_{n-1}}{4(n+\frac{1}{2})(n-\frac{1}{2})+1} = \frac{-3c_{n-1}}{4n^2}.$$

Thus a solution is

$$y_1(x) = |x|^{1/2}\sum_{n=0}^{\infty} c_n x^n, \text{ ただし } c_n = \frac{-3c_{n-1}}{4n^2}, \ n \geq 1. \blacksquare$$

Theorem 4.8 In the 2nd-order linear differential equation

$$y'' + P(x)y' + Q(x)y = 0,$$

if $x = 0$ is the regular singular point and the root r of the indicial equation is multiple root, then the linearly independent solutions y_1 and y_2 are given by the following forms.

$$\begin{aligned} y_1 &= |x|^r \sum_{n=0}^{\infty} c_n x^n \ (c_n \neq 0), \\ y_2 &= y_1 \log|x| + |x|^r \sum_{n=0}^{\infty} C_n x^n. \end{aligned}$$

Now we find y_2. By the theorem, we find a solution of the form

$$y_2 = y_1 \log|x| + |x|^{3/2}\sum_{n=0}^{\infty} C_n x^n$$

For the matter of calculation we set $x > 0$, $c_0 = 1$. Substitute y_2 into $L(y) = 0$.

$$\begin{aligned} L(y_2) &= 4x^2(y_1'' \log x + \frac{2y'}{x} - \frac{y_1}{x^2}) \\ &+ \sum_{n=0}^{\infty} 4(n+\frac{3}{2})(n+\frac{1}{2})C_n x^{n+3/2} + (3x+1)y_1 \log x + \sum_{n=0}^{\infty} 3C_n x^{n+5/2} \\ &+ \sum_{n=0}^{\infty} C_n x^{n+3/2} \\ &= \log x L(y_1) + 8xy' - 4y_1 + 3C_0 x^{3/2} + C_0 x^{3/2} \\ &+ \sum_{n=1}^{\infty}[4(n+\frac{3}{2})(n+\frac{1}{2})C_n + C_n + 3C_{n-1}]x^{n+3/2} = 0 \end{aligned}$$

Now express a few terms of y_1.

$$y_1 = x^{1/2} - \frac{3}{4}x^{3/2} + \frac{9}{64}x^{5/2} - \frac{27}{(36)(64)}x^{7/2} + \cdots.$$

Note that $L(y_1) \equiv 0$. Then

$$(4C_0 - 6)x^{3/2} + (16C_1 + 3C_0 + \frac{9}{4})x^{5/2} + (36C_2 + 3C_1 - \frac{9}{32})x^{7/2} + \cdots = 0$$

From this $C_0 = \frac{3}{2}, C_1 = -\frac{27}{64}, C_2 = \frac{11}{256}, \ldots$. Thus

$$y_2 = \log x(x^{1/2} - \frac{3}{4}x^{3/2} + \frac{9}{64}x^{5/2} - \frac{3}{256}x^{7/2} + \cdots) + \frac{3}{2}x^{3/2} - \frac{27}{64}x^{5/2} + \frac{11}{256}x^{7/2} + \cdots. \blacksquare$$

Example 4.6 Find a solution of the following differential equation

$$L(y) = x^2y'' + (x^2 - 3x)y' + 3y = 0.$$

SOLUTION Since $P(x) = \frac{x^2-3x}{x^2}$, $Q(x) = \frac{3}{x^2}$, $x = 0$ is a regular singular point. Then let $y = \sum_{n=0}^{\infty} c_n x^{n+r}$.

$$\begin{aligned} L(\sum_{n=0}^{\infty} c_n x^{n+r}) &= \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} \\ &- \sum_{n=0}^{\infty} 3(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} 3c_n x^{n+r} = 0. \end{aligned}$$

Let the power of x be put together with x^{n+r}. Then

$$\sum_{n=0}^{\infty}[(n+r)(n+r-1) - 3(n+r) + 3]c_n x^{n+r} + \sum_{n=1}^{\infty}(n+r-1)c_{n-1}x^{n+r} = 0.$$

From this

$$\begin{aligned} &\underbrace{(r(r-1) - 3r + 3)c_0 x^r}_{n=0} \\ + &\sum_{n=1}^{\infty}\{[(n+r)(n+r-1) - 3(n+r) + 3]c_n + (n+r-1)c_{n-1}\}x^{n+r} = 0 \end{aligned}$$

Now the indicial equation $r^2 - 4r + 3 = 0$ implies that $r = 1, 3$. Then we have two solutions.

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+1}, \; y_2(x) = \sum_{n=0}^{\infty} C_n x^{n+3}$$

Put the coefficients of x^{n+r} equal to 0. Then we have the recurrence equation

$$c_n = \frac{-(n+r-1)c_{n-1}}{(n+r)(n+r-1) - 3(n+r) + 3}$$

For $r = 1$, we have

$$c_n = \frac{-nc_{n-1}}{(n+1)n - 3(n+1) + 3} = \frac{-nc_{n-1}}{n^2 - 2n}$$

For $r = 3$, we have

$$C_n = \frac{-(n+2)C_{n-1}}{(n+3)(n+2) - 3(n+3) + 3} = \frac{-(n+2)C_{n-1}}{n^2 + 2n}.$$

Now note that $c_{n+2} = C_n$. Thus $y_1 = y_2$. Then we need to find the independent solution.

Theorem 4.9 For the 2nd-order linear differential equation

$$y'' + P(x)y' + Q(x)y = 0,$$

if $x = 0$ is a regular singular point and the difference of the roots r_1, r_2 is positive integer, then the linearly independent solutions y_1, y_2 are given by the following:

$$y_1 = |x|^{r_1} \sum_{n=0}^{\infty} c_n x^n \ (c_n \neq 0),$$

$$y_2 = c y_1 \log|x| + |x|^{r_2} \sum_{n=0}^{\infty} C_n x^n \ (C_0 \neq 0).$$

Thus $y_2 = c y_1 \log x + x \sum_{n=0}^{\infty} C_n x^n$ can be found by substituting y_2 into $L(y) = 4x^2 y'' + (3x+1)y = 0$.
■

4.3.1 Exercise

1. Find singular points of the following differential equations and classify.
(a) $x^2 y'' + xy' + y = e^x$
(b) $x(x-1)^3 y'' + xy' + (x-1)^2 y = 0$
(c) $(2-x)y'' + xy' + \frac{y}{(x-2)^2} = 0$
2. Find a solution of the following differential equation around $x = 0$.
(a) $2xy'' + 3y' - y = 0$
(b) $xy'' + y' + xy = 0$
(c) $4x^2 y'' - 2x(x-2)y' - (3x+1)y = 0$
3. Find the solutions of the following differential equations around the designated point.
(a) $(1-x^2)y'' - 2xy' + n(n+1)y = 0,\ a = 0$
This equation is called **Legendre's equation**
(b) $(1-x^2)y'' - 2xy' + 12y = 0,\ a = 1$
(c) $x^2 y'' + xy' + (x^2 - \nu^2)y = 0,\ a = 0$
This is called **Bessel's equation**.